Math 523b - Homework 6 (Last one!)

1. Folland problem 22, p. 225
2. Folland problem 24, p. 225
3. Folland problem 27, p. 225
4. In a homework problem you showed that the dual of c_0, the space of sequences that converge to zero with the sup norm, is isometric to l^1. Now let c denoted the space of real-valued sequences that converge to some finite real number. Given $y = \{y_n\}_{n=1}^{\infty} \in l^1$ we can still define a linear functional f_y by

$$f_y(\{x_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} x_n y_n$$

   and the same argument as for c_0 proves this is a bounded linear functional with norm equal to the l^1 norm of y. So we have an isometry from l^1 **into** c^*.

   (a) The Riesz representation theorem says that the dual of c_0 is isometric with a space of measures. Explain briefly how this is related to the assertion that c_0^* is isometric with l^1.

   (b) Prove that the isometry above from l^1 into c^* is not onto.

   (c) Prove that there is a linear function $f_\infty \in c^*$ so that every linear functional in c^* can be written as $f_y + a f_\infty$ for some $y \in l^1$ and some real number a. (So the range of the isometry has co-dimension 1.)

5. Let $f(x)$ be a Borel measurable real-valued function on $[0, 1]$ with $\int_0^1 f^2(x)\,dx < \infty$. Let $\{U_n\}_{n=1}^{\infty}$ be an independent sequence of random variables, each of which is uniformly distributed on $[0, 1]$, i.e., the distribution P_{U_N} of U_n is Lebesgue measure on $[0, 1]$. Let

$$S_n = \frac{1}{n} \sum_{i=1}^{n} f(U_i)$$

   Prove that S_n converges to the constant $\int_0^1 f(x)\,dx$ in probability.

6. Let E_n be a sequence of events. Recall that

$$\limsup E_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$$

In probability this event is usually denoted $E_n i.o.$ where i.o. stands for "infinitely often." Let X_n be an independent, identically distributed sequence of non-negative random variables. Prove that $EX_1 < \infty$ if and only if $P(X_n \geq n \ i.o.) = 0$.

7. Suppose X_n are independent, identically distributed, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

(b) Now suppose that you also know that $EX_n^4 < \infty$. Prove that $P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p.