Math 523b - Take home midterm

Due Monday, March 25 at 10 am
Rules: You may not talk to anyone about the exam. If the problem is unclear you can ask me for clarification, but I will not give hints. You may refer to your class notes, your homework solutions and Folland. But you cannot use any other books or the web.
Exposition is important. I will take off points if your solution is not clearly explained or contains irrelevant arguments.

1. Let $\mathcal{E}$ be the set of even polynomials which we consider as functions on $[0,1]$. For $1 \leq p \leq \infty$ this is a subspace of $L^p([0,1],\mathcal{B},m)$ where m is Lebesgue measure and $\mathcal{B}$ are the Borel sets. Determine (with proof) the closure of $\mathcal{E}$ in L^p for $1 \leq p \leq \infty$.

2. Let $(X,\mathcal{M},\mu)$ be a measure space. We abbreviate $L^p(X,\mathcal{M},\mu)$ as L^p. Let $1 < p, r < \infty$. Fix a function $f \in L^p$ and define a linear operator T by $Tg = fg$, where $g \in L^r$. Find a condition on p, q, r that implies $Tg \in L^q$ for all $g \in L^r$ and prove that T is a bounded linear operator from L^r to L^q when this condition holds.

3. Let $\mathcal{P}$ be the set of power series $\sum_{n=0}^{\infty} c_n x^n$ such that

$$\sum_{n=0}^{\infty} (n+1)|c_n| \leq 1$$

It is easy to show that such a series converges uniformly on $[0,1]$ and so defines a continuous function. So $\mathcal{P} \subset C([0,1])$. (You don't have to show this). Prove that the closure of $\mathcal{P}$ in $C([0,1])$ is compact. The norm is the usual sup norm.

4. Let $1 < p < \infty$ and let q be the conjugate exponent given by $1/p+1/q = 1$. Let b_n be a real valued sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for all real sequences a_n in l^p. Prove that b_n is in l^q. Hint: consider the linear functionals f_N on l^p defined by

$$f_N(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{N} b_n a_n$$

5. Consider $L^2 = L^2([0,\infty), \mathcal{B}, m)$ where m is Lebesgue measure and $\mathcal{B}$ is the Borel sets. Let

$$V_n = \{f \in L^2 : f = 0 \, a.e. \, on \, (n, \infty)\}$$

($f = 0 \, a.e. \, on \, (n,\infty)$ means that $m(\{x \in (n,\infty) : f(x) \neq 0\}) = 0.$) It is not hard to show V_n is closed. You may assume this fact without proving it. Let P_n be the orthogonal projection onto V_n.

(a) Find an explicit formula for P_n, i.e., given $f \in L^2$ what is $P_n f$?

(b) It is natural to expect that P_n converges to the identity I in some sense. Here are two possible senses in which it could converge.

i. $||P_n - I|| \to 0$. (Here the norm is the operator norm.)

ii. P_n converges to I in the strong operator topology, i.e., for all $f \in L^2$, $||P_n f - f|| \to 0$. (Here the norm is the norm in L^2.)

For each of these sense, either prove or disprove that P_n converges to I in this sense.