5 SLE

5.1 Definition via Loewner equation

In this chapter B_t will denote a standard one-dimensional real-valued Brownian motion starting at 0. So $EB_t = 0$ and $EB_t^2 = t$. With probability one, B_t is a continuous function and so theorem 21 applies.

Definition 19 *The chordal SLE with parameter $\kappa \geq 0$ is the random family of conformal maps g_t which are the solution to*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z \tag{141}$$

in the upper half plane when the solution exits.

As before, H_t is the domain of g_t and K_t its complement in $\mathbb{H}$. So H_t is the set of initial conditions for which the solution of the differential equation still exists at time t. And K_t is a random family of increasing compact $\mathbb{H}$-hulls.

If $r > 0$ then $r^{-1} B_{r^2 t}$ is also a standard Brownian motion. This leads to the following.

Proposition 14 *Let $r > 0$ and let g_t be chordal SLE$_\kappa$. Let $\hat{g}_t(z) = r^{-1} g_{r^2 t}(rz)$. Then $\hat{g}_t(z)$ has the distribution of chordal SLE$_\kappa$.*

Proof:

$$\dot{\hat{g}}_t(z) = r \dot{g}_{r^2 t}(rz) = r \frac{2}{g_{r^2 t}(rz) - \sqrt{\kappa} B_{r^2 t}} = \frac{2}{r^{-1} g_{r^2 t}(rz) - \sqrt{\kappa} r^{-1} B_{r^2 t}} = \frac{2}{\hat{g}_t(z) - \sqrt{\kappa} r^{-1} B_{r^2 t}} \tag{142}$$

By the scaling of Brownian motion, $r^{-1} B_{r^2 t}$ is also a standard Brownian motion. ∎

Corollary: $r^{-1} K_{r^2 t}$ has the same distribution as K_t.

It is important to note that the $\sqrt{\kappa}$ cannot be scaled out of the above equation. The solutions and the K_t will change qualitatively as κ changes.

Exercise: Show that the solution to

$$\dot{g}_t(z) = \frac{(2/\kappa)}{g_t(z) - B_t} \tag{143}$$

is SLE$_\kappa$ parameterized so that $hcap(K_t) = 2t/\kappa$.

Theorem 24 *With probability one, SLE$_\kappa$ is generated by a path.*

Definition 20 *The chordal SLE$_\kappa$ path in $\mathbb{H}$ is the random path that generates SLE$_\kappa$.*

The proof of the theorem will come much later. We will also show that

$$\lim_{z \to \gamma(t)} g_t(z) = \sqrt{\kappa} B_t \tag{144}$$

We define SLE in other domains by requiring that it be conformally invariant. There is one slightly subtle point in the chordal case. Let D be a simply connected domain and $z, w \in \partial D$. Let F be a conformal map of D onto $\mathbb{H}$ with $F(z) = 0$, $F(w) = \infty$. Then we define

$$h_t(z) = F^{-1}(g_t(F(z))) \tag{145}$$

This is a conformal map of $F^{-1}(H_t)$ onto D. We define it to be chordal SLE in the domain D from z to w. Note, however, that the conformal map F is not unique. For any $r > 0$, rF is another such conformal map. In fact, all other conformal maps sending z to 0 and w to ∞ are of this form. If we use rF in the above we get

$$\hat{h}_t(z) = F^{-1}(r^{-1} g_t(rF(z))) = F^{-1}(\hat{g}_{t/r^2}(F(z))) \tag{146}$$

Thus we just get a different time parameterization. So the definition is independent of the choice of F up to a time reparameterization. (a non-random one).

Definition 21 *The radial SLE with parameter $\kappa \geq 0$ is the random family of conformal maps g_t which are the solution to*

$$\dot{g}_t(z) = g_t(z) \frac{\exp(i\sqrt{\kappa} B_t) + g_t(z)}{\exp(i\sqrt{\kappa} B_t) - g_t(z)} \quad g_0(z) = z \tag{147}$$

in the unit disc $\mathbb{D}$ when the solution exits.

As before, H_t is the domain of g_t, i.e., is the set of initial conditions in $\mathbb{D}$ for which the solution of the differential equation still exists at time t. K_t its complement in $\mathbb{D}$, and so K_t is a random family of increasing subsets of the disc.

5.2 Derivation of SLE

In the previous section we defined SLE in the chordal case by taking the driving function to be a Brownian motion. In this section we will explain why this is a natural thing to do based on properties of the statistical physics models. This argument for the radial case may be found in Werner's St. Flour notes.

We start by reviewing the conformal invariance property and Markov property we stated in sections 3.6 and 3.7. Suppose we have a random family of simple curves γ which start at the origin and then stay in the upper half plane. We assume that the curves are parameterized so that $hcap(\gamma[0,t]) = 2t$. We suppose that these random curves have both the property of conformal invariance and the Markov property. Together these two properties imply the following property.

Fix a time $T > 0$. We condition on what the curve has done up to time T. Let g be the conformal map of $\mathbb{H} \setminus \gamma[0,T]$ onto $\mathbb{H}$ with the usual normalizations. Consider the image under g of the curve after time T, i.e., $g(\gamma[T, T+t])$. It is a curve that starts at U_T. If we translate it so that it starts at the origin, then the two properties imply that with the conditioning on $\gamma[0,T]$ it has the same distribution as the original random curve and it is independent of $\gamma[0,T]$.

Now take two independent copies of γ, call them γ^1 and γ^2. We let $g_t^i(z)$ be the conformal map of $\mathbb{H} \setminus \gamma^i[0,t]$ onto $\mathbb{H}$ with the usual normalizations. We construct a curve $\hat{\gamma}(t)$ as follows. For $t \leq T$, $\hat{\gamma}(t)$ is just $\gamma^1(t)$. For $t > T$, consider the curve $\gamma^2(t-T)$, but shift it by U_T^1 so that it starts at U_T^1, i.e., consider $\gamma^2(t-T) + U_T^1$. Then look at its pre-image under g_T^1,i.e., look at $\hat{\gamma}(t) = (g_T^1)^{-1}(\gamma^2(t-T) + U_T^1)$. This is a simple curve that starts at $\gamma^1(T)$ and does not intersect the curve $\hat{\gamma}[0,T]$. So $\hat{\gamma}(t)$ defines a simple curve starting at the origin for all $t \geq 0$. The above property says that $\hat{\gamma}(t)$ has the same distribution as $\gamma(t)$.

Let $\hat{g}_t(z)$ be the conformal map of $\mathbb{H} \setminus \hat{\gamma}[0,t]$ onto $\mathbb{H}$ with the usual normalizations. Then $\hat{g}_t(z)$ and $g_t(z)$ have the same distribution. Let $U_t, \hat{U}_t, U_t^1, U_t^2$ be the driving functions of $g_t, \hat{g}_t, g_t^1, g_t^2$. Obviously, $\hat{g}_t = g_t^1$ for $t \leq T$. So $\hat{U}_t = U_t^1$ for $t \leq T$. Now consider $t > T$. We have

$$\hat{g}_t(z) = g_{t-T}^2(g_T^1(z) - U_T) + U_T \tag{148}$$

Letting $\cdot$ denote differentiation with respect to t this implies

$$\dot{\hat{g}}_t(z) = \frac{2}{g_{t-T}^2(g_T^1(z) - U_T^1) - U_t^2} = \frac{2}{\hat{g}_t(z) - U_T^1 - U_t^2} \tag{149}$$

Thus $\hat{U}_t = U_T^1 + U_{t-T}^2$. And so $\hat{U}_t - \hat{U}_T = U_{t-T}^2$. Since U_t and $\hat{U}_t$ have the same distribution, we have shown that the increment $U_t - U_T$ is independent of U_s for $s \leq T$ and the distribution of this increment does not depend on T. Thus the process U_t has independent, stationary increments.

In all our models the random curve γ has a trivial symmetry that we did not bother to mention explicitly, namely, γ and $-\gamma$ have the same distribution. The conformal map that takes the half plane minus the curve $-\gamma[0,t]$ onto the half plane is just $-g_t(z)$. Thus the driving function for $-\gamma(t)$ is $-U_t$. So we conclude that U_t and $-U_t$ have the same distribution. In particular, the mean of U_t is zero for all t.

As a function of t, U_t is continuous. So we have learned that U_t is a continous process with stationary, independent increments and mean zero. Lawler and Werner both assert that this implies that U_t must be a Brownian motion. It has mean zero, but its variance is not fixed. So there is a one free parameter. So we can write U_t as a constant times a standard Brownian motion. The convention is to take this constant to be $\sqrt{\kappa}$, so that $U_t = \sqrt{\kappa} B_t$.