8 Convergence of loop-erased random walk to SLE_2

8.1 Comments on the paper

The proof that the loop-erased random walk converges to SLE_2 is in the paper "Conformal invariance of planar loop-erased random walks and uniform spanning trees" by Lawler, Schramm and Werner. (arxiv.org: math.PR/0112234). This section contains some comments to give you some hope of being able to actually make sense of this paper.

In the first sections of the paper, δ is used to denote the lattice spacing. The main theorem is that as $\delta \to 0$, the loop erased random walk converges to SLE_2. **HOWEVER,** beginning in section 2.2 the lattice spacing is 1, and in section 3.2 the notation δ reappears but means something completely different. (δ also appears in the proof of lemma 2.1 in section 2.1, but its meaning here is restricted to this proof. The δ in this proof has nothing to do with any δ's outside the proof.)

After the statement of the main theorem, there is no mention of the lattice spacing going to zero. This is hidden in conditions on the "inner radius." For a domain D containing 0 the inner radius is

$$rad_0(D) = \inf\{|z| : z \notin D\} \tag{364}$$

Now fix a domain D containing the origin. Let ψ be the conformal map of D onto $\mathbb{U}$. We want to show that the image under ψ of a LERW on a lattice with spacing δ converges to SLE_2 in the unit disc. In the paper they do this in a slightly different way. We take the scaling limit by using the domain aD with $a \to \infty$ and using a unit lattice. Let ψ_a be the conformal map from aD to $\mathbb{U}$. (Of course, $\psi_a(z)$ is just $\psi(z/a)$.) Then the image of the LERW in aD on a unit lattice under ψ_a is trivially the same as the image of the LERW on a lattice of spacing $1/a$ in D. Note that $rad_0(aD) = a\, rad_0(D) \to \infty$. So in the lemmas/theorems in the paper you will find hypotheses of the form "given $\delta > 0$ there is an $r_0(\delta)$ such that for domains D with $rad_0(D) > r_0(\delta)$ the following is true ..." The δ here is not the lattice spacing. In the above, the smaller δ is the larger $r_0(\delta)$ must be. The condition $rad_0(D) > r_0(\delta)$ will actually be applied to aD and so it is really a condition that a is large, i.e., the lattice spacing is small.

t_j is the capacity of $\gamma[0, j]$. This is defined to be the capacity of $\psi(\gamma[0, j])$ in the unit disc. So t_j is the same whether we think of putting a unit lattice in aD or a lattice of spacing $1/a$ in D. As the lattice spacing goes to zero, the spacing between the times t_j goes to zero. The proof of theorem 3.7 uses a subsequence of these times t_{m_n}. The subsequence m_n is defined using a small positive number δ which is not the lattice spacing. Roughly speaking the idea is to divide the time interval into intervals of width δ^2. (We use δ^2 so that over the small time interval the change in a standard Brownian motion will be of order δ.) δ is small but then we make the lattice spacing small enough so that the spacing between times t_j is much smaller than δ^2. The division points $n\delta^2$ will typically not coincide with some t_j. We could define m_n to be the index so that t_{m_n} is closest to

$n\delta^2$. This is not how m_n is defined because it needs to have some some other properties, but it is a good way to think about m_n.

The proof shows that the driving function at time t_{m_n} is very close to $B(2t_{m_n})$ where $B(t)$ is a standard Brownian motion. The times t_{m_n} are very close together, so this and the continuity of Brownian motion shows that the driving function is close to $B(2t)$. Note that $B(2t)$ has the same distribution as $\sqrt{2}B(t)$, the driving function for SLE_2.

8.2 Notation guide

This is a list of the notation used in the paper.

δ is the lattice spacing in the first parts of the paper, but beginning in section 3.2 is is used for something completely different.

$\mathbb{U}$ is the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

D is a simply connected domain containing 0.

Γ is an ordinary random walk from 0 to ∂D.

β is the loop erasure of Γ

γ is the loop erasure of the time reversal of Γ, so γ_0 is on the boundary of D and $\gamma_l = 0$.

$$D_j = D \setminus \cup_{i=0}^{j-1}[\gamma_i, \gamma_{i+1}] = D \setminus \gamma[0, j] \tag{365}$$

ψ is conformal map of D onto $\mathbb{U}$ with $\psi(0) = 0$, $\psi'(0) > 0$.

ψ_j is conformal map of D_j onto $\mathbb{U}$ with $\psi_j(0) = 0$, $\psi_j'(0) > 0$.

t_j is capacity of $\gamma[0, j]$ in D which by definition is the capacity of $\psi(\gamma[0, j])$ in $\mathbb{U}$ which by definition is $\ln(\psi_j'(0))$.

W_t is the driving function of $\psi \circ \gamma$

$\Theta(t)$ is continuous real valued so that $W_t = \exp(i\Theta(t))$

U_j is defined to be $W(t_j)$. This equals $\psi_j(\gamma_j)$

$\Delta_j = \Theta(t_j)$

$m_0 = 0$. Then m_n is defined inductively by

$$m_{n+1} = \min\{j > m_n : |\Delta_j - \Delta_{m_n}| \geq \delta \ or \ |t_j - t_{m_n}| \geq \delta^2\} \tag{366}$$

$\mathcal{F}_n$ is σ-field generated by $\gamma[0, m_n]$.

The martingale M_n is

$$M_n = \sum_{j=0}^{n-1} \left(\Delta_{m_{j+1}} - \Delta_{m_j} - E[\Delta_{m_{j+1}} - \Delta_{m_j} | \mathcal{F}_j]\right) \tag{367}$$