2 Percolation

There is a vast literature on percolation. For the reader who wants more than we give here, there is an entire book: *Percolation*, by Geoffrey Grimmett. A good account of the recent spectacular results on percolation using the Schramm-Lowener evolution is Wendelin Werner's lectures from the summer school in Utah in 2007. (They are in the arxiv.org archive.)

2.1 Definition of the model

Percolation can be defined on any lattice in any number of dimensions. It also comes in two flavors - bond percolation and site percolation. We will start with bond percolation and to be concrete think of the lattice $\mathbb{Z}^d$. We will denote the set of nearest neighbor bonds or edges in the lattice by $\mathbb{E}^d$.

We start with a hand-waving definition of the model and then give the precise definition. We fix a parameter $p \in [0,1]$. Each bond is the lattice is "open" with probability p and "closed" with probability $1-p$. The bonds are independent. So the model generates a random subset of $\mathbb{E}^d$.

Here is the more precise definition. The sample space is

$$\Omega = \prod_{e \in \mathbb{E}^d} \{0,1\}$$

with the convention that 1 represents open and 0 represents closed. $\mathcal{F}$ is the σ-field generated by the finite dimensional cylinders. On each $\{0,1\}$ we put the probability measure that gives 1 probability p and 0 probability $1-p$. Then we take P to be the product probability measure.

spell out what this means for an event that only depends on a finite number of bonds

We use ω to denote a point in the sample space. This corresponds to a function $\omega(e)$ from $\mathbb{E}^d$ into $\{0,1\}$. We can represent such a function by the set of edges e with $\omega(e)=1$, which we will refer to as a bond configuration. Given such a bond configuration we let C denote the connected component which contains the origin. C can contain an infinite or finite number of bonds. Let $|C|$ denote the number of bonds in C. We define

$$\theta(p) = P(|C| = \infty)$$

So $\theta(p)$ is the probability that the origin belongs to an infinite cluster. It is trivial that

$$\theta(p) = 1 - \sum_{n=0}^{\infty} P(|C| = n)$$

Going beyond the question of whether there is an infinite cluster, we might ask what such a large scale structure looks like. We have defined the model on a lattice with unit lattice spacing. A very interesting thing to do is define it on a lattice with spacing a, e.g., $a\mathbb{Z}^d$ and studying the scaling limit or continuum limit in which $a \to 0$. We can do this in a finite volume. A particular interesting question is the following. Take a rectangle of size L by M. Put in a lattice of spacing a. Then we ask if there is a cluster of open bonds that connects the top edge to the bottom edge. Similarly, is there a cluster of open edges that connects the right edge to the left edge. In the finite volume these probabilities will lie in $(0, 1)$. The interesting question is what happens in the scaling limit.

So far we have been considering bond percolation. There is another model called site percolation. Again, we fix a parameter $p \in [0, 1]$. Each site is open with probability p and closed with probability $1-p$. Again we study connected cluster, but now the definition of connected is slightly different. A set of sites in the lattice is connected if we can get from any one site in the set to any other site in the set by a nearest neighbor path which only visits sites in the set. We refer to connected sets of sites as clusters and now C denotes the cluster containing the origin. $\theta(p)$ is defined as above, and the questions about crossings of a rectangle in the scaling limit are just as interesting.

2.2 Proof of a phase transition

It is not hard to show that $\theta(p) < 1$ if $p < 1$. (We leave this as an easy exercise.)

Proposition 1 *$\theta(p)$ is non-decreasing, $\theta(0) = 0$, and $\theta(1) = 1$.*

Proof: The fact that it is non-decreasing is intuitively obvious. To prove it we use a powerful idea known as "coupling." Explain what this is in general.

We define another probability space. We put i.i.d. random variables's X_e on the edges with each X_e uniformly distributed on $[0, 1]$. More precisely, let Ω' be the product

$$\Omega' = \prod_e [0, 1] \tag{1}$$

Now fix a p. Given an outcome in Ω', i.e., values of the X_e, we define a bond configuration $\omega_p(e)$ by declaring a bond to be open if $X_e \leq p$. This puts the same probability measure on the set of bond configurations as before.

Now suppose $p_1 < p_2$. Then $\omega_{p_1} \leq \omega_{p_2}$ is the sense that for every edge $\omega_{p_1}(e) \leq \omega_{p_2}(e)$.

If ω_{p_1} has an infinite cluster containing the origin, then so does ω_{p_2}. Hence the probability that there is an infinite cluster for p_1 is smaller than the probability there is an infinite cluster for p_2, i.e., $\theta(p_1) \leq \theta(p_2)$.

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We define a critical value by $p_c = \sup\{p : \theta(p) = 0\}$. The monotonicity of $\theta(p)$ then implies $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. But we can't say whether or not

$\theta(p_c)$ is zero. In one dimension $p_c = 1$ (exercise). In two and higher dimensions there is a phase transition, as we will now prove.

Theorem 1 *For $d \geq 2$, $0 < p_c < 1$.*

Proof: We first prove that $p_c > 0$. So we want to show that for sufficiently small p, $\theta(p) = 0$. For a finite connected set D of bonds which contains the origin, we define E_D to be the event that all the bonds in D are open. Clearly $P(E_D) = p^{|D|}$. We showed in the previous chapter that there is lattice dependent constant c such that the number of such D with n bonds is bounded by c^n. The key idea is that if the connected component containing the origin is infinite then for any n it contains a finite connected subset with n bonds that contains the origin. So for any n,

$$\{|C| = \infty\} \subset \cup_{connected\, D\, containing\, 0,\, |D|=n} E_D \tag{2}$$

So

$$P(\{|C| = \infty\}) \leq \sum_{connected\, D\, containing\, 0,\, |D|=n} P(E_D) \leq c^n p^n \tag{3}$$

If $p < 1/c$ this bound goes to 0 as $n \to \infty$. So $P(\{|C| = \infty\}) = 0$.

Now we show that $p_c < 1$. So we must show that if p is close to 1 there is a nonzero probability that there is an infinite cluster. We use a Peierls argument. We define the dual lattice as before. We then define a edge configuration on the dual lattice as follows Note that each edge in the original lattice is bisected by one edge in the dual lattice. This sets up a one to one correpondence between edges in the original lattice and edges in the dual lattice. Given an edge configuration for the original lattice we define a edge in the dual lattice to be open if and only if the edge in the original lattice that it bisects is open. Clearly the resulting configurations of edges in the dual lattice is a percolation process on the dual with the same p.

Now suppose the cluster containing the origin is finite. Then we can find a loop of closed edges in the dual lattice which encloses the origin. (It is highly nontrivial to write out an honest proof of this. We prove it by drawing pictures.)

Let γ be a loop of edges in the dual lattice, and let E_γ be the event that all the edges in γ are closed. Then the event that the cluster in the original lattice containing the origin is finite is contained in $\cup_\gamma E_\gamma$. So

$$P(|C| < \infty) \leq \sum_\gamma P(E_\gamma) = \sum_\gamma (1-p)^{|\gamma|}$$

As in the last chapter there is a lattice dependent constant c such that the number of γ containing the origin with n edges is bounded by c^n. So the above is

$$\leq \sum_{n=1}^{\infty} c^n (1-p)^n$$

If p is sufficiently close to 1 this series converges and is less than 1. So for p is sufficiently close to 1, $P(|C| = \infty) > 0$.

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We leave it to the reader to modify these proofs for the case of site percolation. One certainly expects the critical p to depend on the lattice and there is no reason to expect it to be the same for bond and site percolation on the same lattice. For the square lattice you can prove $p_c = 1/2$ for bond percolation. For site percolation on the square lattice the value of p_c is not know exactly but is around 0.59.

Theorem 2 *The probability there is an infinite open cluster somewhere in the lattice is* 0 *if* $\theta(p) = 0$ *and* 1 *if* $\theta(p) > 0$.

Proof

review tail events and the 0-1 law

Let E be the event that there is an infinite open cluster. Note that whether or not there is an infinite open cluster does not change if we change the bond configuration on a finite number of bonds. So E is a tail event. By the 0-1 law it has probability 0 or 1.

If $\theta(p) = 0$ then the probability the origin belongs to an infinite cluster is 0. Let C_x be the event that there is an infinite cluster containing x. By translation invariance, for all x, $P(C_x) = 0$. The event that there is an infinite cluster somewhere is contained in $\cup_x C_x$. So it has probability 0 since there are a countable number of terms in this union.

Now suppose that $\theta(p) > 0$. So the probability there is an infinite cluster containing the origin is not zero. Hence the probabilty there is an infinite cluster somewhere is not zero. But this probability is 0 or 1, so it must be 1. ■

2.3 Critical exponents, universality

Just as for the Ising model there behavior of percolation at and near the critical point is described by several critical exponents.

It is believed that $\theta(p)$ goes to zero as $p \to p_c$ and in the manner

$$\theta(p) \asymp (p - p_c)^\beta, p \to p_c^+ \tag{4}$$

This statement should be taken to mean that the following limit exists and is finite and nonzero:

$$\lim_{p \to p_c^+} \frac{\log(\theta(p))}{\log(p - p_c)} \tag{5}$$

If we are in the *subcritical* phase, $p < p_c$ then the cluster containing the origin is finite a.s. Define

$$\chi(p) = E[|C|] = \infty P(|C| = \infty) + \sum_{n=0}^{\infty} n P(|C| = n) \tag{6}$$

In the subcritical phase the first term on the right side is zero, but it is unclear whether the second term is finite. In fact it is, but it diverges as $p \to p_c$.

$$\chi(p) \asymp (p_c - p)^{-\gamma}, p \to p_c^- \tag{7}$$

In the supercritical phase, $p > p_c$, $\chi(p)$ is simply infinity. But we can look at a "truncated" version:

$$\chi^f(p) = E[|C| 1(|C| < \infty)] = \sum_{n=0}^{\infty} n P(|C| = n) \tag{8}$$

It is also believed to behave as

$$\chi^f(p) \asymp (p - p_c)^{-\gamma}, p \to p_c^+ \tag{9}$$

with the same exponent.

In the subcritical phase we have already asserted that $P(|C| = n)$ decays fast enough that $E[|C|]$ is finite. In fact it decays exponentially fast:

$$P(|C| = n) \asymp \exp^{-\alpha(p)n} \tag{10}$$

At the critical point it decays as a power. The power is usually defined using the cumulative distribution ($|C| \geq n$ rather than $|C| = n$) :

$$P(|C| \geq n) \asymp n^{-1/\delta} \tag{11}$$

which should be taken to mean

$$-\frac{1}{\delta} = \lim n \to \infty \frac{log(P(|C| \geq n))}{\log(n)} \tag{12}$$

Instead of looking at the number of edges in the cluster containing the origin, one can look at the size of this cluster. We let $r(C)$ be the radius of the cluster, the distance to the site in the cluster farthest from the origin. Then

$$P(n \leq r(C) < \infty) \asymp n^{-1/\delta_r} \tag{13}$$

One can define a correlation length by

$$\xi^2(p) = \frac{1}{\chi^2(p)} \sum_x |x|^2 \, P(\{0 \to x\} \cap \{|C| < \infty\}) \tag{14}$$

and it should diverge as

$$\xi(p) \asymp |p - p_c|^{-\nu} \tag{15}$$

At the critical point,

$$P(0 \to x) \asymp |x|^{2-d-\eta} \tag{16}$$

There are scaling relations that are believed to hold in any number of dimensions

$$\begin{aligned} \gamma + 2\beta &= \beta(\delta+1) \\ \gamma &= \nu(2-\eta) \end{aligned} \tag{17}$$

and hyperscaling relations that should only hold for $2 \leq d \leq 6$.

$$\begin{aligned} d\delta_r &= \delta+1 \\ 2-\eta &= d\frac{\delta-1}{\delta+1} \end{aligned} \tag{18}$$

Theorem 3 *For site percolation on the triangular lattice,*

$$\begin{aligned} &\beta = 5/36, \quad \gamma = 43/18, \quad \nu = 4/3 \\ &\eta = 5/24, \quad \delta_r = 5/48, \quad \delta = 91/5 \end{aligned} \tag{19}$$

The theorem should be true for site or bond percolation on any two dimensional lattice.

We now return to the question raised previously about when there is a crossing (a path of open bonds) between the sides of a box.

We take a rectangle with with L and height M, and introduce a lattice with spacing a. We consider the probability that there is a crossing between the top and bottom sides. For $0 < p < 1$ this probability is always strictly between 0 and 1. The interesting question is what happens when $a \to 0$. If $p < p_c$ it can be proved that the probability goes to 0, and if $p > p_c$ it can be proved that it goes to 1. At $p = p_c$ it converes to a value strictly between 0 and 1. In fact there is an explicit formula for this probability in terms of L/M. (It is left as an exercise to show that it can only depend on L and M through their ratio L/M.) This formula was first found by John Cardy using conformal field theory and then proved by Stas Smirnov for site percolation on the triangular lattice. The formula should be true for bond or site percolation on any two dimensonal lattice, but only this one case has been proved.

In fact Cardy's formula is much more general. Take any domain and take two "arcs" along its boundary. Cardy's formula gives the probability there is a crossing between the two boundary arcs.

Exercise 2.1 (easy) Prove that $\theta(p) < 1$ if $p < 1$. What can you say about the behavior of $\theta(p)$ as $p \to 1^-$.

Exercise 2.2 (easy) Prove that in one dimension, $\theta(p) = 0$ for $p < 1$.

Exercise 2.3 (non-rigorous). Consider the following percolation model. We take the hexagonal (or honeycomb) lattice. Each hexagon is open with probability p and closed with probability $1-p$.
(a) Show that this is the same as site percolation on the triangular lattice. (Hint: think about dual lattices.)
(b) Take a large square centered at the origin. We impose "mixed boundary conditions" as follows. For hexagons along the boundary whose center has a non-negative x coordinate we declare the hexagon to be open, for boundary hexagons whose center has negative x coordinate we declare it to be closed. There will be a vertical edge along the bottom boundary that has an open boundary hexagon on one side and a closed boundary hexagon on the other side. There will be a similar vertical edge in the top boundary. Show there is a unique path of edges between these two edges with the property that as we traverse the path from bottom to top, the hexagons immediately to the left of the path are all closed and those immediately to the right are all open. We will call this path the interface.
(c) What does the interface look like is p is very close to 0? What if p is very close to 1?
(d) Show that if we replace p by $1-p$ and reflect about the y-axis, the model is unchanged. Using this symmetry make a guess for the critical value of p.

Exercise 2.1.4 (trivial in the end) Take a rectangle with dimensions L by M. Assume L/M is rational. We introduce a lattice with spacing a and do percolation and study the probability that there is a crossing (path of open bonds) between the top and bottom edges. Assume this probability has a limit as $a \to 0$. Denote it by $p(L, M)$. Use only the assumption that the limit as $a \to 0$ exists to prove that $p(L, M)$ only depends on L/M.

Exercise 2.5 (long, but the basis for lots of important stuff) The goal of this exercise is to derive a nice representation of the Ising model as a correlated cluster model. This representation, known as the Fortuin Kasteleyn representation, is the basis for various rigorous results and for fast Monte Carlo algorithms for the Ising model. Write the partition function for the Ising model in the form

$$Z = \sum_{\sigma} \exp[\beta \sum_{<ij>} (\sigma_i \sigma_j + 1)] \tag{20}$$

(This differs from the original definition by a factor of $\exp(\beta N_e)$ where N_e is the number of edges. Let $\delta_{\sigma_i,\sigma_j}$ be the function that is 1 if $\sigma_i = \sigma_j$ and is 0 if $\sigma_i \neq \sigma_j$. Then we have the identity

$$\exp[\beta(\sigma_i \sigma_j + 1)] = 1 + (e^{2\beta} - 1)\delta_{\sigma_i,\sigma_j} \tag{21}$$

Use this to show that

$$Z = \sum_B (e^{2\beta} - 1)^{|B|} 2^{N(B)} \tag{22}$$

where B is summed over all subsets of the set of bonds and $N(B)$ is the number of connected components of B. Define p by

$$\frac{p}{(1-p)} = e^{2\beta} - 1 \tag{23}$$

Show that if we threw out the factor of $2^{N(B)}$ in the above it would be essentially the same as percolation with p given by the above formula.