4 Random walks

4.1 Simple random walk

We start with the simplest random walk. Take the lattice $\mathbb{Z}^d$. We start at the origin. At each time step we pick one of the $2d$ nearest neighbors at random (with equal probability) and move there. We continue this process and let $S_m \in \mathbb{Z}^d$ be our position at time m.

Here is a more careful definition. Let X_k be a sequence of random vectors taking values in $\mathbb{Z}^d$ which are independent. Each X_k takes on the $2d$ values $\pm e_i, \quad i = 1, 2, \cdots, d$ with probability $1/2d$ where e_i is the unit vector in the ith direction. Then we define

$$S_m = \sum_{k=1}^{m} X_k \tag{1}$$

Note that the quantities in this sum are vectors.

How far do we travel after m steps? Since $E[X_k] = 0$, we have $E[S_m] = 0$. So the average position of the walk is always the origin. (This is just a trivial consequence of the symmetry.) To compute the distance we could consider $E[|S_m|]$ where $|\quad|$ denotes the length of the vector. But it is much easier to compute the mean squared distance travelled:

$$E[S_m^2] = \sum_{k=1}^{m} \sum_{l=1}^{m} E[X_k \cdot X_l] \tag{2}$$

If $k \neq l$, then by the independence $E[X_k \cdot X_l] = E[X_k] \cdot E[X_l] = 0$. If $k = l$, $E[X_k \cdot X_l] = E[1] = 1$. So $E[S_m^2] = m$. So the root mean squared distance behaves as $E[S_m^2]^{1/2} = m^\nu$ with $\nu = 1/2$. The exponent ν can be thought of as a critical exponent. It is a bit strange to be talking about critical phenomena here. Usually in statistical mechanics one must tune at least one parameter to make the system critical. We will return to this point later.

Now we generalize the model. Instead of the nearest neighbor walk we allow it to make more general jumps. So X_k is a sequence of independent, indentically distributed random variables with values in Z^d. The only constraint we keep is that $E[X_k] = 0$. (Note that X_k is a vector and 0 is the zero vector here.) The above calculation still works and we have

$$E[S_m^2]^{1/2} = cm^{1/2} \tag{3}$$

where $c^2 = E[X_k \cdot X_k]$. In other words $\nu = 1/2$ for a wide class of random walks. We don't need to stay on the lattice. We can let the X_k take values in $\mathbb{R}^d$ and get a walk in the continuum (although time is still discrete).

The S_m form a discrete time stochastic process. We make this into a continuous time stochastic process by linear interpolation. More precisely,

$$S_t = \begin{cases} S_t & \text{if } t \text{ is an integer} \\ \textit{linear on } [\text{m,m+1}] & \text{if } t \in [m, m+1] \end{cases} \tag{4}$$

The typical size of S_t is $\sqrt{t}$ which motiviates the following rescaling. For each positive integer n, we let

$$S_t^n = n^{-1/2} S_{nt} \tag{5}$$

For $d = 1$, if we picture a graph of S_t, then to get S_t^n we shrink the horizontal (time) axis by a factor of n and shrink the vertical (space) axis by a factor of $\sqrt{n}$. Note that for t which are equal to an integer divided by n, the variance of S_t^n is t.

The scaling limit is obtained by letting $n \to \infty$. The result is Brownian motion. In the next section we define Brownian motion and give a precise statement of the result that the scaling limit of the random walk is Brownian motion

4.2 Brownian Motion

This discussion follows two books: Chapter 7 of *Probability: Theory and Examples* by Richard Durrett and chapter 2 of *Brownian Motion and Stochastic Calculus* by Ioannis Karatzas and Steven Shreve.

We recall a basic construction from probability theory. Let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e., a measure space with $P(\Omega) = 1$. Let $X_1, X_2, \cdots, X_m$ be random variables, i.e., measurable functions. Then we can define a Borel measure μ on $\mathbb{R}^m$ by

$$\mu(B) = P((X_1, X_2, \cdots, X_m) \in B) \tag{6}$$

where B is a Borel subset of $\mathbb{R}^m$. One can then prove that for a fuction $f(x_1, x_2, \cdots, x_m)$ which is integrable with respect to μ, we have

$$Ef(X_1, X_2, \cdots, X_m) = \int_{\mathbb{R}^m} f(x_1, x_2, \cdots, x_m) d\mu \tag{7}$$

Of course, this measure depends on the random variables; when we need to make this explicit we will write it as $\mu_{X_1,\cdots,X_n}$.

The random variables $X_1, X_2, \cdots, X_m$ are said to be independent if the measure $\mu_{X_1,\cdots,X_n}$ equals the product of the measures $\mu_{X_1}, \mu_{X_2}, \cdots \mu_{X_m}$. Two collections of random variables $(X_1, \cdots, X_m)$ and $(Y_1, \cdots, Y_m)$ are said to be equal in distribution if $\mu_{X_1,\cdots,X_n} = \mu_{Y_1,\cdots,Y_n}$.

We now turn to Brownian motion. It is a continuous time stochastic process. This means that it is a collection of random variables X_t indexed by a real paramter t.

Definition 1. *A one-dimensional (real valued) Brownian motion is a stochastic process B_t, $t \geq 0$, with the following properties.*
(i) If $t_0 < t_1 < t_2 < \cdots t_n$, then B_{t_0}, $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$ are independent random variables.
(ii) If $s, t \geq 0$, then $B_{t+s} - B_s$ has a normal distribution with mean zero and variance t. So

$$P(B_{t+s} - B_s \in A) = \int_A (2\pi t)^{-1/2} \exp(-x^2/2t) dx \tag{8}$$

where A is a Borel subset of the reals.
(iii) With probability one, $t \to B_t$ is continuous.

In short, Brownian motion is a stochastic process whose increments are independent, stationary and normal, and whose sample paths are continuous. Increments refer to the random variables of the form $B_{t+s} - B_s$. Stationary means that the distribution of this random variable is independent of s. Independent increments means that increments corresponding to time intervals that do not overlap are independent. Proving that such a process exists is not trivial, but we will not give the proof. The above definition makes no mention of the underlying probability space Ω. One can take it to be the set of continuous functions $\omega(t)$ from $[0, \infty)$ to $\mathbb{R}$ with $\omega(0) = 0$. Then the random variables are given by $B_t(\omega) = \omega(t)$. Unless otherwise stated, we will take $B_0 = 0$. We list some standard consequences of the above properties.

Theorem 1. *If B_t is a Brownian motion then*
(a) B_t is a Gaussian process, i.e., for any times $t_1, \cdots, t_n$, the distribution of $B_{t_1}, \cdots, B_{t_n}$ has a multivariate normal distribution.
(b) $EB_t = 0$ and $EB_sB_t = \min\{s, t\}$.

(c) Define

$$p(t,x,y) = (2\pi t)^{-1/2} \exp(\frac{-(x-y)^2}{2t}) \tag{9}$$

Then for Borel subsets $A_1, A_2, \cdots, A_n$ *of* $\mathbb{R}$*,*

$$P(B_{t_1} \in A_1, B_{t_2} \in A_2, \cdots, B_{t_n} \in A_n) =$$
$$\int_{A_1} dx_1 \int_{A_2} dx_2 \cdots \int_{A_n} dx_n \quad p(t_1, 0, x_1)\, p(t_2 - t_1, x_1, x_2) \cdots, p(t_n - t_{n-1}, x_{n-1}, x_n)$$

Exercise: Prove the above. Hint for (b): If random variables X and Y are independent, then $E\,XY = EX\,EY$. For $s > t$, write B_s as $(B_s - B_t + B_t)$.

The definition of d-dimensional Brownian motion is easy. We take d independent copies of one-dimensional Brownian motion, and label them as $B_t^1, B_t^2, \cdots, B_t^d$. Then $(B_t^1, B_t^2, \cdots, B_t^d)$ is a d-dimensional Brownian motion. We can also think of the two-dimensional Brownian motion (B_t^1, B_t^2) as a complex valued Brownian motion by considering $B_t^1 + iB_t^2$.

The paths of Brownian motion are continuous functions, but they are rather rough. With probability one, the Brownian path is not differentiable at any point. If $\gamma < 1/2$, then with probability one the path is Hölder continuous with exponent γ. But if $\gamma > 1/2$, then the path is not Hölder continuous with exponent γ. For any interval (a, b), with probability one the path is neither increasing or decreasing on (a, b). With probability one the path does not have bounded variation. This last fact is important because it says that one cannot use the Riemann-Stieltjes integral to define integration with respect to B_t.

For later purposes we make the following observation. Suppose we only look at Brownian motion at integer times: B_n. Define $X_k = B_k - B_{k-1}$. Then X_k is independent and each X_k has a standard normal distribution. So $B_n = \sum_{k=1}^n X_k$ is random walk with Gaussian steps.

4.3 Brownian motion as scaling limit of random walks

We now return to the process defined by rescaling the random walk, eq (5). We take $d = 1$ and assume that $E[X_k^2] = 1$. Consider times $0 < t_1 < t_2 < \cdots t_m$ where each time is equal to some integer divided by n. (**should replace** n **by** 2^n **???**) Consider the random variables $S_{t_1}, S_{t_2} - S_{t_1}, \cdots S_{t_m} - S_{t_{m-1}}$. Each of them is a sum of a subset of the X_i and no X_i appears in more than one of these sums. Thus these random variables are independent.

If n is large, each of the random variables is the sum of a large number of i.i.d. random variables and so is approximately normal. So S_t^n is looking like Brownian motion, at least at the times which are multiples of $1/n$. So we can hope that as $n \to \infty$, S_t^n will converge to Brownian motion. This is indeed a theorem, proved by Donsker in 1951 and sometimes called the *invariance principle*. To state it in its strongest form requires a definition about convergence of measures. We start by stating a weaker form that is a bit easier to digest.

Theorem 2. *(invariance principle) Fix times $0 < t_1 < t_2 < \cdots < t_m$. We use E^{rw} to denote expectation with respect to the probability measure for the original i.i.d. sequence X_i. Let X_t be a Brownian motion. We use E^{bm} to denote expectation with respect to its probability measure. Then for every bounded continuous function $f(x_1, x_2, \cdots, x_m)$ on $\mathbb{R}^m$, we have*

$$\lim_{n\to\infty} E^{rw} f(S_{t_1}^n, S_{t_2}^n, \cdots, S_{t_m}^n) = E^{bm} f(X_{t_1}, X_{t_2}, \cdots, X_{t_m}) \tag{10}$$

This is already a pretty good theorem and the following somewhat technical discussion is only to get a stronger statement of the above and can be skipped without a big loss. The technical stuff ends where we consider how Brownian motion illustrates the ideas of scaling limits, critical phenomena and universality.

Definition 2. *Suppose that the sample space Ω is a metric space. Suppose that P_n is a sequence of probability measures on Ω defined on the Borel subsets. Let P be another such probability measure. We say that P_n converges weakly to P if*

$$\lim_{n\to\infty} \int f dP_n = \int f dP \tag{11}$$

for every bounded, continuous real-valued function f on Ω.

Now look at the conclusion of the theorem. For each n let μ_n be the probability measure on $\mathbb{R}^m$ that comes from the random variables $S_{t_1}^n, S_{t_2}^n, \cdots, S_{t_m}^n$. Let μ be the probability measure on $\mathbb{R}^m$ that comes from $X_{t_1}, X_{t_2}, \cdots, X_{t_m}$. Then the conclusion of the above theorem is that μ_n converges weakly to μ. A probabilist says that the sequence of random vectors $(S_{t_1}^n, S_{t_2}^n, \cdots, S_{t_m}^n)$ *converges in distribution* to $(X_{t_1}, X_{t_2}, \cdots, X_{t_m})$. And the conclusion of the above theorem is that the finite dimensional distributions of S_t^n converge in distribution to those of Brownian motion.

The stronger form of the theorem does not just look at the process at a finite set of times. Let $C[0,\infty)$ be the space of continuous functions on $[0,\infty)$. We let P denote the probability measure on this space for Brownian motion. For each n, S_t^n is a continuous function of t. So S_t^n also defines a probability measure on $C[0,\infty)$. We denote it by P_n. It is supported on piecewise linear functions.

Theorem 3. *(Invariance principle of Donsker) Let X_i be an i.i.d. sequence of random varibles defined on the probability space $(\Omega, \mathcal{F}, P)$. Suppose that they have mean zero and variance 1. Define S_t^n by the linear interpolation and scaling defined above, and let P_n be the probabilty measure on $C[0,\infty)$ induced by the process S_t^n. Then P_n converges weakly to a probability measure P for which $B_t(\omega) = \omega(t)$ is standard one-dimensional Brownian motion.*

What about higher dimensions? There is an easy extension. Take $X_k = (X_k^1, X_k^2, \cdots, X_k^d)$ where the full set of X_k^i $k = 1, 2, 3, \cdots, i = 1, 2, \cdots, d$ is independent and we assume $E[X_k^i] = 0$ and $E[(X_k^i)^2] = 1$. Then is follows immediately from the one-dimensional result that S_t^n converges to a d dimensional Brownian motion in the same sense as the $1d$ theorem.

If we consider X_k which do not have independent components, things are a little more involved. Here is a silly example. Let X_k^1 be independent, taking on the values ± 1 with probability $1/2$. Then define $X_k^2 = X_k^1$. The resulting random walk stays on the line with slope 1. It does not converges to $2d$ Brownian motion. (In fact it will converge to a $1d$ Brownian motion with modified variance.) Back to the general situation. For d dimensional Brownian motion, we have

$$E[B_t^i B_t^j] = \delta_{i,j} t \tag{12}$$

So if the random walk is to have a chance of converging to Brownian motion we need

$$E[X_k^i X_k^j] = \delta_{i,j} \tag{13}$$

and of course $E[X_k^i] = 0$. This is in fact sufficient to get convergence to d dimensional Brownian motion. If (13) does not hold, we will get convergence to what you might call a correlated Brownian motion in which

$$E[B_t^i B_t^j] = C_{i,j} t \tag{14}$$

where the matrix C is given by

$$C_{i,j} = E[X_k^i X_k^j] \tag{15}$$

We now consider how Brownian motion illustrates the ideas of scaling limits, critical phenomena and universality. We start with the scaling limit. Usually in statistical physics one starts with a model defined on a lattice and then tries to understand what the scaling limit is. If we take $X_i = \pm 1$ with equal probability, then the random walk stays on the lattice $\mathbb{Z}$. The scaling limit is what we did above when we shrunk time by a factor of n and space by a factor of $\sqrt{n}$. For this model we have a candidate for the scaling limit (Brownian motion) and a theorem that says the scaling limit is indeed equal to Brownian motion. This is not the typical situation in statistical physics. There we are lucky if we have an explicit candidate for the scaling limit and extremely lucky if we have a theorem that says the scaling limit does coverge to the candidate.

Now consider universality. The invariance principle is a very strong form of universality. It says that we can start with any random walk, subject only to the conditions that the steps have mean zero and variance 1, and the scaling limit will converge to the same stochastic process, i.e., Brownian motion. We have stated the invariance principle only for one dimension. But it is true in any number of dimensions. For example, we can take a random walk on the lattice $\mathbb{Z}^d$ which at each step moves by $\pm e_i$ with probability $1/2d$ where e_i is the unit vector in the ith coordinate direction. We then take a scaling limit as we did above. This will converge to a d-dimensional Brownian motion. (I am ignoring a slight rescaling that needs to be done here.)

Finally we consider criticality. In the scaling limit the steps of the random walk are of size $1/\sqrt{n}$. So the random walk is formed by combining infinitely many microscopic random inputs. The result, Brownian motion, is clearly random. So it appears that Brownian motion is a critical phenomena. This is a bit confusing from the viewpoint of statistical physics. Usually in a statistical physics model one must adjust a parameter, e.g., the temperature, to a particular value to make the model have critical behavior. There appears to be no such parameter in the random walk model. To see how the random walk is critical we must consider it as a special case of a more general model. We give two ways to doing this. The first is rather simple, but the second is more interesting and more relevant for what we will do with the self-avoiding walk.

In some sense the condition that the mean of the step X_i must be zero plays the role of adjusting a parameter to make the model critical. Consider a one-dimensional random walk with steps of ± 1, but now take $X_i = 1$ with probability p and $X_i = -1$ with probability $1 - p$ with $p \neq 1/2$. Now the typical size of S_n is n, not $\sqrt{n}$ as before. So to construct a scaling limit we must define

$$S_t^n = n^{-1} S_{nt} \tag{16}$$

Now in the scaling limit, S_t^n will converge to a straight line with a slope which depends only on p. So the scaling limit has no randomness at all. Thus the microscopic randomness produces macroscopic randomness only at the critical point $p = 1/2$.

The second way of generalizing the random walk is the following. For concreteness we work in two dimensions on the square lattice but you can do this in any dimension on any lattice. Fix a domain containing the origin, e.g., a unit disc centered at the origin. Introduce a lattice with spacing $1/n$. Note that we use $1/n$ rather than $1/\sqrt{n}$. Now run the walk until it first exits the disc. The result is a probability measure on nearest neighbor walks ω that start at the origin and end on the boundary or just outside the disc. Note that these walks have varying length which we will denote by $|\omega|$. The probability of a single ω is $4^{-|\omega|}$. The scaling limit is given by letting $n \to \infty$. It gives a probabilty measure on curves in the domain that start at the origin and end on the boundary. The scaling limit is equal to the probability measure we get by starting a Brownian motion at the origin and running it until it exits the domain.

Now we generalize the model. We take all nearest neighbor walks that start at the origin and end just outside the domain and give such a curve the weight $e^{-\beta|\omega|}$. Then we normalize the resulting measure. If we take $e^\beta = 4$, this given the original random walk. For larger values of β we can think of it as the original random walk model with a penalty based on the length of the walk. Longer walks are supressed. Suppose β is really large. Then the probability measure will be dominated by the shortest walks from the origin to the boundary. So the microscopic randomness only shows up at the macroscopic scale in a trivial way. As we lower β this will continue to be the case until we reach the critical value of $\beta = \ln 4$ when we see macroscopic randomness in the scaling limit.

Exercise: For $p \neq 1/2$, find the slope m of the line to which (16) converges.

Prove that for $t > 0$,

$$\lim_{n\to\infty} S_t^n = mt \tag{17}$$

with probability one. Hint: law of large numbers.

Exercise: Consider the nearest neighbor simple random walk on the square lattice. So X_k takes on the values $(1,0), (-1,0), (0,1), (0,-1)$, all with probability $1/4$. The components of X_k are not independent. Now suppose we rotate the square lattice by 45 degrees. We still consider the nearest neighbor walk, so the steps are along lines with slope 1 or -1. Show that X_k now has independent components and so we can conclude that the scaling limit is a two dimensional Brownian motion.

Exercise: Consider the model of nearest neighbor walks in a domain that start at the origin and end on the boundary of the domain weighted by $e^{-\beta|\omega|}$. For concreteness consider the walk on the square lattice, so the critical value of β is $\ln(4)$. What happens to the model if $\beta < \ln(4)$? Hint: first consider the extreme case of $\beta = 0$ and compute the normalizing factor for the probability measure.

4.4 Self-avoiding random walk

We take a lattice, e.g., in two dimensions the square, triangular or hexagonal lattice, and we fix a natural number N. We consider all walks with N steps which start at the origin, take only nearest neighbor steps and do not visit any site more than once. So a walk ω is a function ω from $\{0, 1, 2, \cdots, N\}$ into the lattice such that

$$\begin{gathered} \omega(0) = 0 \\ |\omega(i) - \omega(i-1)| = 1, \quad i = 1, 2, \cdots N \\ \omega(i) \neq \omega(j), \quad 0 \leq i < j \leq N \end{gathered} \tag{18}$$

There are a finite number of such walks for any fixed N, and we put a probability measure on this set by requiring that all such walks be equally probable.

The self-avoiding walk is of interest to physicists since it is model for polymers in dilute solution. More generally, it is of interest since it is a simple model that exhibits critical phenomena and universality. There are a variety

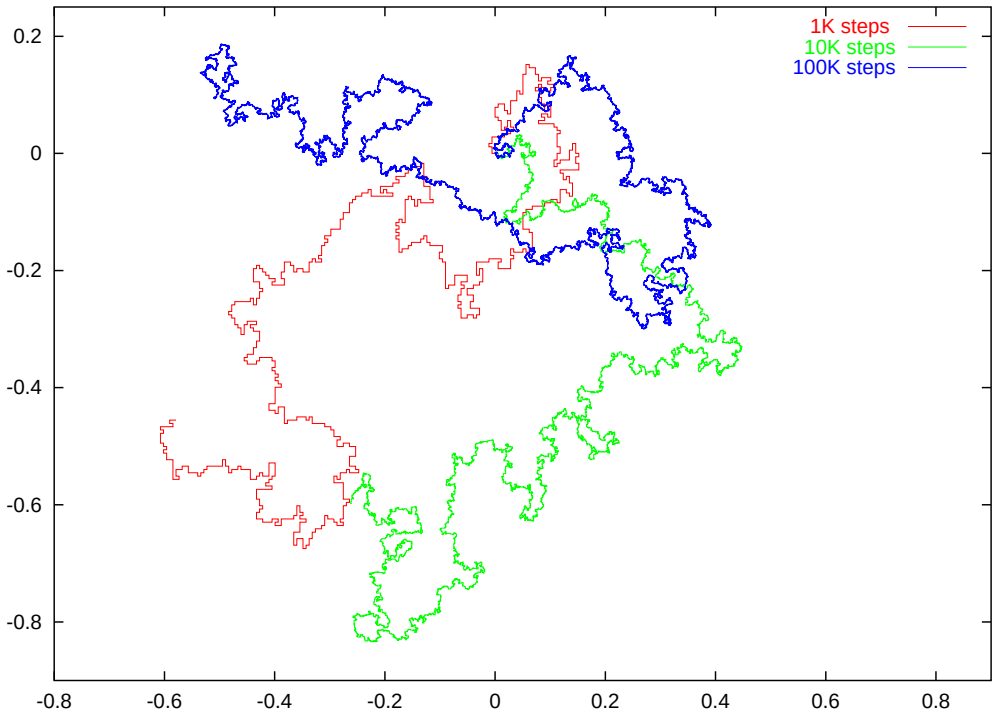

Figure 1: Three self-avoiding walks in the full plane with 1K, 10K and 100K steps. Each walk has been scaled by $N^{-3/4}$.

of critical exponents that describe the behavior of the model. Figure 1 shows three self-avoiding walks with $N = 1,000$, $N = 10,000$ and $N = 100,000$. Each walk has been scaled by $N^{-3/4}$ so that they are all on a scale of order one.

Let c_N be The number of self-avoiding walks with a given length N is a very hard problem and no one expects an explicit answer. We can say something about c_N. We claim $c_{n+m} \leq c_n c_m$. (The proof is left as a homework.) This implies $\lim \ln(c_N)/\ln(N)$ exists. Call it $\ln(\mu)$. For the square lattice numerical works says $\mu \approx 2.638$. On the hexagonal lattice there is a conjecture that $\mu = \sqrt{2+\sqrt{2}}$.

It is believed that

$$c_N \asymp \mu^N N^{\gamma-1} \tag{19}$$

where μ depends on the particular lattice (and of course on the number of dimensions) but γ only depends on the number of dimensions. At first it looks like γ is a really uninteresting exponent since in the above it describes a small correction to the geometric growth of the number of SAW's. But it also describes something a lot more interesting. Suppose we take two N step SAW's starting at the origin and ask what is the probability that they don't intersect. If they don't then together they form a $2N$ step SAW. It does not start at the origin, but it has its midpoint at the origin. So we will create all $2N$ step SAW's with their midpoint at the origin this way. Hence the probability the two N steps SAW's do not intersect must be

$$\frac{C_{2N}}{c_N^2} = \frac{(2N)^{\gamma-1}\mu^{2N}}{[N^{\gamma-1}\mu^N]^2} = \frac{2^{\gamma-1}}{N^{\gamma-1}} \tag{20}$$

So the prob is proportional to $N^{1-\gamma}$.

Another critical exponent is related to the growth of the mean distance the walk travels as a funtion of N. The critical exponent ν is defined by

$$E[\omega(N)^2] \sim N^{2\nu} \tag{21}$$

The expected value E is with respect to the uniform probability measure described above and $\omega(N)^2$ is the square of the distance from the origin to the lattice point $\omega(N)$. Everyone believes that in two dimensions $\nu = 3/4$. There are essentially no rigorous results on ν. In fact, there is not even a proof in two (or three) dimensions that ν is bigger than $1/2$, the value for the ordinary random walk.

The exponent ν does not tell us how the endpoint of the walk is distributed, so the next quantity to look at is the distribution of $\omega(N)$. It is natural to scale it by a factor of $N^{-\nu}$ and study this distribution in the limit that N goes to ∞. Of course, for the ordinary random walk this would give a Gaussian distribution. The limiting distribution for the self-avoiding walk is not expected to be Guassian, but is still expected to be rotationally symmetric, so we will only look at the distribution of $N^{-\nu}|\omega(N)|$. This distribution for the self-avoiding walk is shown in figure 2. We also show the distribution for the distance from the endpoint of the walk to its midpoint. We have scaled both random variables so that they each have mean equal to 1. For comparison the analogous distribution for the ordinary random walk is also shown.

A third critical exponent may be defined as follows. Consider all SAW with N steps that start at the origin but then stay in the upper half plane.

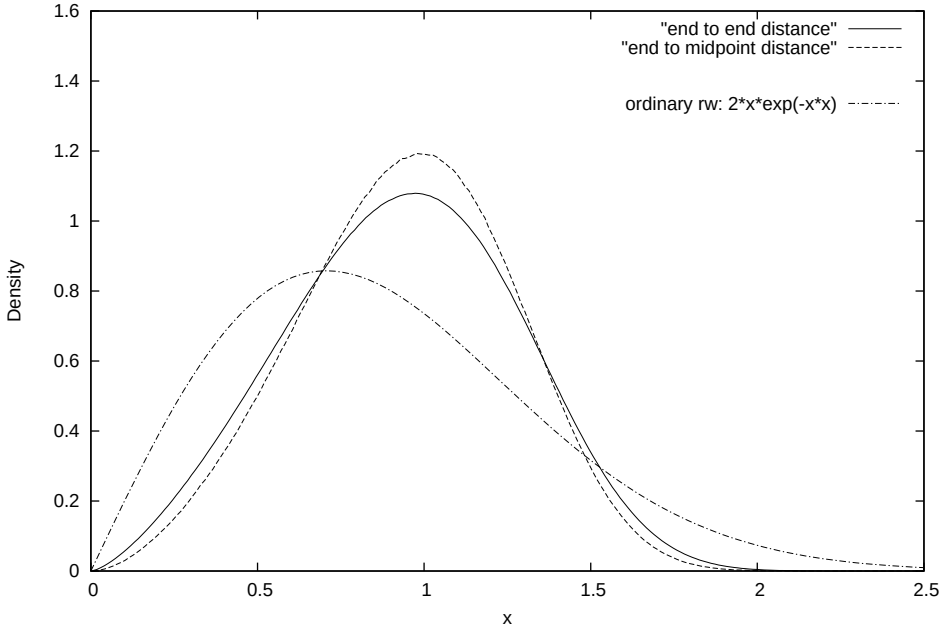

Figure 2: Distribution of the end to end distance and the end to midpoint distance for the SAW. The distances have been rescaled so their mean is one. The end to end distance for the ordinary random walk is also show.

Let B_N be the number of such SAW's with N steps. It is believed that this number has the same geometric growth as the number of all SAW, but with a different power law :

$$B_N \asymp \mu^N N^{\gamma-1-\rho} \tag{22}$$

Here γ is the same as before. So the above defines the exponent ρ. The reason for this way of setting up the exponent is that the probability that an N step SAW starting at the origin stays in the upper half plane is

$$\frac{B_N}{A_N} \asymp \frac{\mu^N N^{\gamma-1-\rho}}{\mu^N N^{\gamma-1}} = N^{-\rho} \tag{23}$$

The above definition was in the full plane. Now let D be a connected unbounded domain with $0 \in \partial D$. An important example is the upper half plane. Introduce the lattice $\delta\mathbb{Z}^2$. Fix an integer N and consider all the

self-avoiding nearest neighbor walks that start at the origin and stay inside D. We put a probability measure on this finite set by making these walks equally probable. Now we take two limits. First we send $N \to \infty$. This should give a probability measure on infinite self-avoiding walks that stay in D. Everyone believes this limit exists, but this has only been proved for the half-plane. Next we let $\delta \to 0$. This should give a probability measure on continuous curves in D that start at 0 and presumably "end" at ∞. This defines the scaling limit of the self-avoiding walk in an unbounded domain between a boundary point and ∞. There are no rigorous results on the existence of this limit. Note that ∞ can be thought of as a boundary point for an unbounded domain.

Now consider a bounded domain D and let z and w be points on its boundary. We want to define the SAW in D between these two points. Again, we introduce the lattice $\delta\mathbb{Z}^2$. We take all self-avoiding walks that go from z to w and stay in D. We now consider walks with any number of steps. Define a probability measure on this finite set by requiring the probability of a walk to be proportional to $e^{-\beta|\omega|}$ where β is a parameter and $|\omega|$ denotes the number of steps in ω. Note that if we do this without the self-avoiding constraint and take $e^{\beta} = 4$ (for the square lattice), then we get the random walk in D starting at z and conditioned to exit D at w. This suggests that for the self-avoiding walk we should choose β as follows. The number of self-avoiding walks of length N is believed to grow like μ^N. We should take $e^{\beta} = \mu$. To construct the scaling limit of the self-avoiding walk we let $\delta \to 0$. (The definition of the scaling limit for a bounded domain is rather different than for an unbounded domain. In particular it only involves a single limit.)

For the Ising model and percolation we got critical behavior only when a parameter (or two) is equal to a specific value. Similarily, we get critical behavior for the self-avoiding walk in a bounded domain only if $\beta = \beta_c$. What happens if $\beta \neq \beta_c$? My guess is that for $\beta > \beta_c$, the scaling limit will give a curve that is just a straight line from z to w (if such a line lies in D.) For $\beta < \beta_c$???

We end this section with a brief discussion of some other versions of the self-avoiding walk. In the model we have considered, we forbid the walk to visit a site more than once. So we could refer to this as the site avoiding walk. Another model is the "bond avoiding walk." We allow all nearest neighbor walks which contain any given bond at most once. Then we put the uniform measure on the set of such walks with N steps. Note that this allows some walks with loops. In fact you can have a really big loop. However, it is

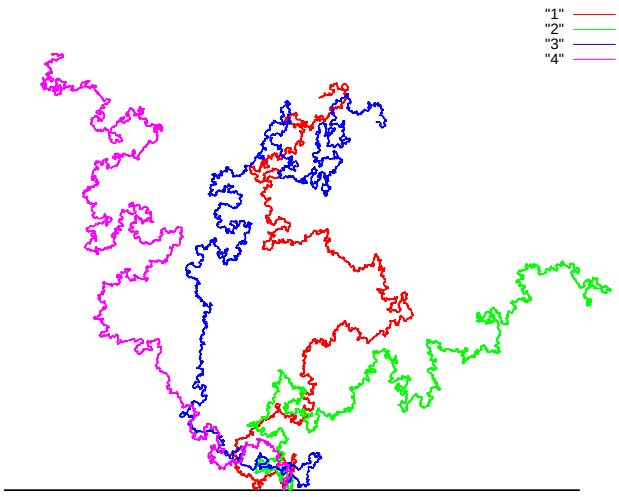

Figure 3: Two of the walks shown are site-avoiding, two are bond-avoiding.

believed that the scaling limit of this model is the same as the scaling limit of the site avoiding walk. Figure 3 shows two bond-avoiding walks, two site-avoiding walks. **Explain why large loops are suppressed.**

Another model is the "weakly self-avoiding walk." We allow all nearest neighbor walks. We take the probability of a walk ω to be proportional to $\exp(-\beta I(\omega))$, where $I(\omega)$ is the number self intersections, i.e., the number of pairs i, j with $0 \le i < j \le N$ such that $\omega(i) = \omega(j)$. It is also believed that the scaling limit of this model is the same as the first self-avoiding walk we defined. In particular, in five and more dimensions the scaling limit of this model has been proved to be Brownian motion. In fact, this is the first model for which there were rigorous results for $d > 4$.

Discuss conformal invariance

Exercise: Recall that c_n is the number of SAW's with n steps which start at the origin. Prove that $c_{n+m} \le c_n c_m$. This says that $\ln(c_n)$ is a subadditive function. Use this to prove

$$\lim_{n\to\infty} \frac{\ln(c_n)}{n} \tag{24}$$

exists and equals $\inf_n \ln(c_n)/n$.

4.5 RG view of the CLT and the random walk

We return to the simple random walk introduced in the first section. For reasons that will be soon be obvious, we take the number of steps to be a power of two, 2^m. We include a scaling factor of $2^{-m/2}$.

$$S_{2^m} = 2^{-m/2} \sum_{k=1}^{2^m} X_k^0 \tag{25}$$

We group the sum as

$$S_{2^m} = 2^{-(m-1)/2} \sum_{k=1}^{2^{m-1}} \frac{X_{2k-1}^0 + X_{2k}^0}{\sqrt{2}} = 2^{-(m-1)/2} \sum_{k=1}^{2^{m-1}} X_k^1 \tag{26}$$

where

$$X_k^1 = \frac{X_{2k-1}^0 + X_{2k}^0}{\sqrt{2}} \tag{27}$$

Note that we added a superscript 0 to the original random variables. In general we will use a superscript m for quantities that are obtained after m interations of the renormalization group. Now we continue.

$$S_{2^m} = 2^{-(m-2)/2} \sum_{k=1}^{2^{m-2}} \frac{X_{2k-1}^1 + X_{2k}^1}{\sqrt{2}} = 2^{-(m-2)/2} \sum_{k=1}^{2^{m-2}} X_k^2 \tag{28}$$

where

$$X_k^2 = \frac{X_{2k-1}^1 + X_{2k}^1}{\sqrt{2}} \tag{29}$$

In general,

$$S_{2^m} = 2^{-(m-p)/2} \sum_{k=1}^{2^{m-p}} \frac{X_{2k-1}^p + X_{2k}^p}{\sqrt{2}} = 2^{-(m-p-1)/2} \sum_{k=1}^{2^{m-p}} X_k^{p+1} \tag{30}$$

where

$$X_k^{p+1} = \frac{X_{2k-1}^p + X_{2k}^p}{\sqrt{2}} \tag{31}$$

Thus we want to study the map on probability distributions for real valued random variables that is given by the following prescription. Let X_1, X_2 be independent and identically distributed. Define

$$X = \frac{1}{\sqrt{2}}[X_1 + X_2] \tag{32}$$

In the following we let $F_Y(y)$ denote the cumulative distribution function of a random variable Y, i.e., $F_Y(y) = P(Y \leq y)$. We assume that all the random variables have continuous distributions, and let f_Y be the density of Y. So f_Y is the derivative of F_Y. Then we have

$$F_X(x) = P(X \leq x) = P(X_1 + X_2 \leq \sqrt{2}x) = F_{X_1+X_2}(\sqrt{2}x) \tag{33}$$

and so

$$f_X(x) = \sqrt{2} f_{X_1+X_2}(\sqrt{2}x) \tag{34}$$

The density of the sum of two independent random variables is the convolution of their densities, so

$$f_{X_1+X_2}(x) = \int f_{X_1}(x-y)\, f_{X_2}(y)\, dy \tag{35}$$

and so

$$f_X(x) = \sqrt{2} \int f_{X_1}(\sqrt{2}x-y)\, f_{X_2}(y)\, dy \tag{36}$$

Since X_1 and X_2 have the same distribution, $f_{X_1} = f_{X_2}$. Denote this common density by $f_0(x)$. So we want to study the map $f_0 \to f_1$ where

$$f_1(x) = \sqrt{2} \int f_0(\sqrt{2}x-y)\, f_0(y)\, dy \tag{37}$$

Since f_0 is a probability density, its integral is 1. It is easy to check that f_1 has this property as well. (It must; it is the density of a random variable.) Likewise f_0 satisfies

$$\int x\, f_0(x) dx = 0 \tag{38}$$

$$\int x^2\, f_0(x) dx = 1 \tag{39}$$

and it is easy to check that f_1 has these properties as well. Again, this also follows immediately from probability considerations.

It is easier to study the map (37) in Fourier space. We let $\hat{f}(k)$ denote the Fourier transform of f:

$$\hat{f}(k) = \int e^{-ikx} f(x)\, dx \tag{40}$$

Then the RG map becomes

$$\hat{f}_1(k) = \hat{f}_0(\frac{k}{\sqrt{2}})^2 \tag{41}$$

It is easy to check that $\exp(-k^2\sigma^2/2)$ is a fixed point of this map for any choice of σ. This is the fourier transform of the normal distribution with mean zero and variance σ^2. The three "conservation laws" become

$$\begin{aligned} f_1(0) &= 1, \\ f_1'(0) &= 0, \\ f_1''(0) &= -1 \end{aligned}$$

What about the stability of this fixed point? We take $\sigma = 1$ for convenience. We want to study the linearization of this map. We consider a pertubation of the form

$$\hat{f}(k) = e^{-k^2/2}[1 + p(k)] \tag{42}$$

We only consider pertubations consistent with the conservation laws. This means

$$p(0) = 0, \quad p'(0) = 0, \quad p''(0) = 0 \tag{43}$$

In other words the Taylor series of $p(k)$ vanished to second order.

Linearized map is

$$(Lp)(k) = 2p(\frac{k}{\sqrt{2}}) \tag{44}$$

Let $p_m(k) = k^m$, then p_m is an eigenfunction with eigenvalue $2^{1-m/2}$. For $m > 2$ this eigenvalue is less than 1. So these are stable directions for the fixed point.

Exercise: Instead of (45) consider

$$X = \frac{1}{2}[X_1 + X_2] \tag{45}$$

We still take X_1 and X_2 to be independent and identically distributed. The only change is that the scaling factor is 2, not $\sqrt{2}$. Show that $\hat{f}(k) = \exp(-|k|)$ is a fixed point of this map. Study the linear stability of this fixed point. What probability density does this correspond to? Why does this not contradict the central limit theorem?

Exercise: Generalize the previous exercise. Hint : $\hat{f}(k) = \exp(-|k|^\alpha)$.