

## 5 Gaussian integral and processes

### 5.1 Jointly Gaussian random variables

**Def** A RV  $X$  is Gaussian if its density is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad (1)$$

We have  $EX = \mu$  and  $\text{var}(X) = \sigma^2$ . The characteristic function (fourier transform) is

$$Ee^{itX} = \exp[it\mu - \frac{1}{2}\sigma^2 t^2] \quad (2)$$

We want to generalize this to  $n$  RV's. It is not enough that each  $X_i$  be Gaussian. We need to specify their joint distribution. This is best done in terms of the characteristic function.

**Def**  $X_1, X_2, \dots, X_n$  are jointly Gaussian if

$$E \exp\left[i \sum_{j=1}^n t_j X_j\right] = \exp\left[i \sum_{j=1}^n t_j \mu_j - \frac{1}{2} \sum_{j,k=1}^n t_j t_k C_{j,k}\right]$$

for some real numbers  $\mu_j$  and a real, non-negative symmetric matrix  $C_{i,j}$ .

**Proposition 1.** (a) Let  $X_1, \dots, X_n$  be jointly Gaussian and  $M_{i,j}$  an  $m \times n$  matrix. Define

$$Y_i = \sum_{j=1}^n M_{i,j} X_j, \quad i = 1, 2, \dots, n$$

Then the  $Y_1, \dots, Y_m$  are jointly Gaussian

(b)

$$E(X_j - \mu_j)(X_k - \mu_k) = C_{jk}$$

(c) If the matrix  $C_{i,j}$  is invertible, then the joint density is

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(C)}} \exp\left(-\frac{1}{2} \sum_{j,k=1}^n (x_j - \mu_j)(x_k - \mu_k)(C^{-1})_{j,k}\right)$$

Here  $C_{j,k}^{-1}$  is the  $j, k$  entry of the inverse of  $C$ .

**Def** Let  $X_t$  be a stochastic process where  $t$  ranges over some index set  $T$ . ( $T$  could be  $\mathbb{R}$ ,  $[0, \infty)$ ,  $\mathbb{Z}^d$ ,  $\mathbb{R}^d$ .) We say  $X_t$  is Gaussian if for all  $n$  and  $t_1, t_2, \dots, t_n \in T$ , the random variables  $X_{t_1}, X_{t_2}, \dots, X_{t_n}$  are jointly Gaussian. The mean of the process is

$$\mu(t) = EX_t$$

The covariance is

$$C(t, s) = EX_t X_s - EX_t EX_s$$

**Explain how mean and covariance completely determine a Gaussian process**

### Exercises

1. Prove part (c) of the proposition.
2. Let  $X_1, X_2, \dots, X_n$  be jointly Gaussian. Recall that the covariance of  $X$  and  $Y$  is  $Covar(X, Y) = E[XY] - E[X]E[Y]$ . Prove that  $X_1, \dots, X_n$  is independent if and only if  $Covar(X_i, X_j) = 0$  for  $i \neq j$ .
3. Let  $B_t$  be standard Brownian motion. Fix a time  $T$  and some real number  $a$ . Let

$$X_t = at + \frac{T-t}{T} B\left(\frac{tT}{T-t}\right) \tag{3}$$

Show that  $X_t$  is a Gaussian process on  $[0, T]$  and compute its mean and covariance functions. This is straightforward. The point of this problem is to introduce this process. It is called the Brownian bridge. Note that the sample paths start at 0 and end at  $a$ . It is in fact what you get if you take a Brownian motion starting at 0 and condition it to end at  $a$  at time  $T$ . For a challenge, prove this. Note that we are conditioning on an event of probability zero, so you have to do this by a limiting process. First condition on the event that it ends within distance  $\delta$  of  $a$  at time  $T$  and then let  $\delta \rightarrow 0$ .

## 5.2 Existence of infinite dimensional Gaussian processes

This may never be written

## 5.3 Examples

### 5.3.1 Brownian motion

Brownian motion is a Gaussian process. Its mean  $\mu(t)$  is identically zero and its covariance is  $C(s, t) = \min\{s, t\}$ .

### 5.3.2 Massless free field

Ising can be rewritten as

$$Z = \sum_{\sigma} \exp[-\beta/2 \sum_{i,j} (\sigma_i - \sigma_j)^2]$$

We continue to work with a lattice, but now we take variables  $\sigma_i$  at each lattice site which take values in  $\mathbb{R}$  instead of  $\pm 1$ . We write them as  $\phi_i$ . So

$$Z = \int \exp[-\beta/2 \sum_{i,j} (\phi_i - \phi_j)^2] d\phi$$

where  $d\phi$  stands for  $\prod_i d\phi_i$ , i.e., the product of Lebesgue measure at each lattice site. This is not defined unless we have a finite number of lattice sites, but the above looks like a Gaussian integral. Pretending that it makes sense, we will try to figure out what the covariance of Gaussian process should be.

$$\sum_{i,j} (\phi_i - \phi_j)^2 = (\phi, -\Delta\phi) \tag{4}$$

Here  $\phi$  stands for the vector  $(\phi_i)_{i \in \Lambda}$  and the inner product on the right side is ... This equation defines the lattice Laplacian operator  $\Delta$ . If we work on the infinite lattice like  $\mathbb{Z}^d$  it is an infinite dimensional matrix with entries

$$-\Delta(i, i) = 2d \tag{5}$$

$$-\Delta(i, j) = -1, \quad \text{if } |i - j| = 1 \tag{6}$$

$$-\Delta(i, j) = 0, \quad \text{if } |i - j| > 1 \tag{7}$$

So the above looks like a Gaussian process with covariance  $(-\Delta)^{-1}$ . To make sense of this we need to study the lattice Laplacian and in particular diagonalize it.

We work on a hypercube

$$\Lambda = \{(j_1, j_2, \dots, j_d) : 1 \leq j_l \leq L\} \quad (8)$$

We define

$$\Lambda^* = \left\{ \left( \frac{2\pi k_1}{L}, \frac{2\pi k_2}{L}, \dots, \frac{2\pi k_d}{L} \right) : 0 \leq k_j \leq L \right\} \quad (9)$$

Then for  $k \in \Lambda^*$ , define

$$e_k(j) = \exp(ik \cdot j) \quad (10)$$

Then

$$-\Delta e_k = \lambda(k) e_k \quad (11)$$

where

$$\lambda(k) = \sum_{j=1}^d 2(1 - \cos(k_j)) \quad (12)$$

### 5.3.3 Orstein-Uhlenbeck process and the massive free field

Covariance is  $(-\Delta + m^2)^{-1}$

#### Exercise

1. *Diagonalizing lattice laplacian*

(a) Show that  $e_k(j)$  is indeed an eigenfunction of the lattice Laplacian with the given eigenvalue.

(b) If you want to drive yourself crazy, consider the lattice Laplacian on the hexagonal and or triangular lattices and find their eigenfunctions and eigenvalues.

2. The covariance of the massive free field on the lattice  $\mathbb{Z}^d$  is

$$C(l, m) = c \int_{[0, 2\pi]^d} d^d k \exp(ik \cdot (l - m)) \left[ \sum_{j=1}^d 2(1 - \cos(k_j)) + m^2 \right]^{-1} \quad (13)$$

Prove that this equation defined a bounded, continuous nonnegative definite function in any number of dimensions. So there is a Gaussian process with this as its covariance.