7 Euclidean field theory as classical statistical mechanics and more RG

7.1 "Spins" in $\mathbb{R}$

In the Ising model the spins at each site take on only two values. There are a variety of ways to generalize this. One can let the spin take on some finite number of values greater than 2 (Potts models). One can let the spin equal points on a circle or sphere (rotator or sigma models). In this chapter we will consider models in which the "spins" s_i take on all real values and the Hamiltonian is of the form

$$H(s) = \frac{1}{2} \sum_{<i,j>} (s_i - s_j)^2 + \sum_i V(s_i)$$

where $V(x)$ is just a real valued function of a single real variable. If s_i only equals ± 1, then $(s_i - s_j)^2 = 2 - 2s_i s_j$, which is just the nearest neighbor Ising interaction.

There are several motivations for studying models like this. First, if we take $V(x)$ to be a "double well" potential like that shown in figure ?????, then the model acts very much like the Ising model since the probability measure is dominated by configurations in which the spins s_i are near the bottoms of the two wells at ± 1. If we replace the quadratic term in the Hamiltonian by a more general quadratic function, it is possible to write the Ising model exactly as a continuous spin system. The transformation that does this is known as ???. We will return to it later. Other models of interest in statistical mechanics, in particular Coulomb systems and dipole systems, may be related to continuous spin systems through a transformation known as the sine-Gordon transformation.

A second motivation comes from the renormalization group itself. All the renormalization group transformations we considered in the previous chapter mapped Ising like models into Ising-like models, i.e., the block spins only took on the values ± 1. Another transformation would be to define the block spin to be the sum of the spins in the block. For 2 by 2 blocks this block spin would take on the values $-4, -2, 0, 2, 4$. As we iterate the transformation the number of possible values of the block spins would grow. Like the quasilinear transformations for the Ising model, this transfomation must also include a rescaling of the spin if it is to have a fixed point. Then in the limit of

infinitely many iterations the block spin would take on all real values, and we would have a continuous spin system. Of course, we would not expect the renormalized Hamiltonian to be so simple that it would be of the above form.

A third motivation for studying such systems is that they are models from a different area of physics - quantum field theory which describes elementary particles. The above model is too simple to describe any real particles, but the theories of electromagnetic interactions and the other interactions between elementary particles are generalizations of the above.

It is customary to denote the continuous spins as ϕ_i rather than s_i and to refer to them as fields rather than spins. As in the case of the Ising models, we need to first define the model in a finite volume. To be concrete we will take the potential $V(x)$ to be $\lambda(x-1)^2(x+1)^2$ where λ is a large parameter. If the purpose of this term is to make our model look like the Ising model, then it is natural to only multiply the interaction $(\phi_i - \phi_j)^2$ by β. So we let

$$H(\phi) = \frac{\beta}{2} \sum_{<i,j>} (\phi_i - \phi_j)^2 + \lambda \sum_i (\phi_i - 1)^2(\phi_i + 1)^2 \tag{1}$$

Let Λ be a finite subset of $\mathbb{R}^d$. The partition function or normalization is given by

$$Z = \int_{\mathbb{R}^\Lambda} \exp[-H(\phi)] \prod_{i \in \Lambda} d\phi_i$$

and we define a probability measure on $\mathbb{R}^\Lambda$ by

$$d\mu(\phi) = Z^{-1} \exp[-H(\phi)] \prod_{i \in \Lambda} d\phi_i$$

We will denote the expectation of a function $F(\phi)$ with respect to such a probability measure by $< F(\phi) >$.

As with the Ising model there are a variety of choices for the boundary conditions. Free boundary conditions mean that we only sum over nearest neighbor pairs $< i, j >$ for which both of i and j are in Λ. We could also sum over all pairs such that at least one of the sites is in Λ and then fix the value of ϕ_i outside the volume. We might fix it to be $+1$ or -1 to try to pick out one of the two wells. We could also fix the value outside the volume to be zero. This is known as Dirichlet boundary conditions. (The first term in the Hamiltonian is a finite difference Laplacian, and the boundary condition

we just described is the analog of the Dirichelt boundary condition for the usual Laplacian.) Periodic boundary conditions may also be defined in the obvious way.

We now fix λ to be some large value. The probability measure will be then be concentrated on configurations for which most of the ϕ_i are close to -1 or $+1$. What happens as we vary β from small to large values? In particular, can we obtain different infinite volume measures by different choices of the boundary conditions.

If β is small we might hope to mimic the proof of ???. We don't have an identity like ??, but we could try something similar by writing

$$\exp[-\frac{\beta}{2}\sum_{<i,j>}(\phi_i-\phi_j)^2] = \prod_{<i,j>}[1+K(\phi_i-\phi_j)]$$

where $K(\phi_i-\phi_j) = \exp[\frac{-\beta}{2}(\phi_i-\phi_j)^2]-1$. We then expand out this product over nearest neighbor bonds to produce a sum of subsets of the set of bonds. Some of the neat tricks we used in the Ising case are no longer present. In particular, the contribution of a particular set of bonds may be nonzero even though the set of bonds has nontrivial boundary. In the Ising proof we had factors of $\tanh(\beta)$ which are small if β is small. In the present case small β implies $K(\phi_i-\phi_j)$ is small if ϕ_i and ϕ_j are of order 1. Unfortunately these fields can be arbitrarily large. This happens with small probability, but the reader can see that things are a good bit more complicated. We will need to deal with the rare events of large fields by a separate argument. Although the argument is more complicated, it is possible to prove that when β is small the boundary conditions don't matter. The model is in a "high temperature" phase.

When β is large we might try to adapt the Peierls argument to the present case. We could define the contour by looking only at the sign of the fields ϕ_i. We would include a bond in the dual lattice in the Peierls contour if the fields on opposite sides of the bond had different signs. Since the fields are usually near ± 1, the energy in the Hamiltonian associated with a contour is usually close to twice the number of bonds in the the Hamiltonian. However, it is possible for the fields on opposite sides of a bond in the contour to both be close to zero. Thus the energy of the contour can be essentially zero. This bad case should be very rare, but clearly it will take a lot more work to carry out the Peierls argument in this case. It can be done, and one can prove that for large β the boundary conditions do matter and the model is in a low

temperature phase.

Like the Ising model the above Hamiltonian has a symmetry. If we change the sign of all the fields, i.e., let $\phi \to -\phi$ then $H(\phi)$ is unchanged. Not all of the boundary conditions we described respect this symmetry. We can also break the symmetry as we did in the Ising model by adding a magnetic field term $h \sum_i \phi_i$ to the model. Another way to generate different infinite volume measure when β is large would be to let $h \to 0^+$ or $h \to 0^-$ after we had taken the infinite volume limit.

As β goes from small to large values there should be a critical value β_c at which the boundary conditions start to matter. (The critical value will depend on the value of λ.) We can study all of the question we did in the Ising model. In particular we can define critical exponents. It is believed that the critical exponents of our continuous spin model are exactly the same as for the Ising model! The second motivation we gave for studying the continuous spin models is the reason. This sort of RG transformation should take Ising models into continuous spin models.

As in the Ising systems, an important object to study is the "two point correlation function ", $< \phi_i \phi_j >$. In particular we would like to know how it decays as $|i - j| \to \infty$. Most of the time it should decay exponentially, and the correlation length ξ is defined by

$$< \phi_i; \phi_j >=< \phi_i \phi_j > - < \phi_i >< \phi_j > \sim e^{-|i-j|/\xi}$$

The Hamiltonian we have been considering above can be written as

$$H(\phi) = \frac{\beta}{2} \sum_{<i,j>} (\phi_i - \phi_j)^2 + \sum_i (\frac{a}{2}\phi_i^2 + \lambda \phi_i^4)$$

with $a = -4\lambda$. More generally we could consider this model for any choice of β, a and λ. While it looks as if we have three parameters, we should keep in mind that in the continuous model we can do a trivial change of variables, $\phi_i \to t\phi_i$, and so there are really only two parameters in the model. Two models which are related by such a change of variables are essentially the same model. In particular they have the same correlation length.

If λ is zero, then the Hamiltonian is just a quadratic function of ϕ. So the integral is a Gaussian integral. For nonzero λ we can write the integral as a perturbation of a Gaussian integral. Let ϕ_j be a Gaussian process with covariance

$$\int \phi_j \phi_k \, d\mu = C(j, k)$$

where $C = (-\beta\Delta + a)^{-1}$. Then the partition function is equivalent to

$$Z = \int \exp(-\lambda \sum_j \phi_j^4)\, d\mu$$

One can write down a continuum version of this. First we are going to change our notation somewhat. The continuum version of our interaction is

$$\frac{\beta}{2} \int (\nabla\phi)^2(x) d^d x + \int [a\phi(x)^2 + \lambda\phi(x)^4] d^d x$$

where $\phi(x)$ is a function on $\mathbb{R}^d$ rather than the lattice. In quantum field theory one wants to study these continuum models, and so it is common even in the lattice case to denote the sites by x and the value of the field at x by $\phi(x)$. The operators in continuum case often have integral kernels, so instead of writing matrix elements as $C_{x,y}$ we will denote them by $C(x, y)$.

Exercise: Consider the Hamiltonian (1) with λ and β large. Impose boundary conditions in which the fields outside the finite volume are fixed to be $+1$. Carry out the Peierls argument to prove the expectation of the field at the origin is close to 1, uniformly in the volume.

7.2 Renormalization group transformations

Renormalization group transformations may be defined for the continuous spin models that are somewhat analagous to those we defined for the Ising type models. For example, we could divide the lattice into 2 by 2 blocks and define the block field for a block to be the sum of the four fields in the block. The new interaction for the block spins is then obtained by integrating over all configurations of the original field that are consistent with a fixed block spin configuration. Label the blocks by α, the block fields by ψ_α and the original fields by $\phi_{\alpha,i}$ where i labels the sites within a block. Formally we would have

$$\exp[-H_1(\psi)] = \int \prod_\alpha \delta(\psi_\alpha - z \sum_i \phi_{\alpha,i}) \exp[-H(\phi)] D\phi$$

where $D\phi$ stands for the product of a copy of Lebesgue meaure for each field $\phi_{\alpha,i}$. Note that we have included a parameter z in the δ function. This

rescaling of the field ("field strength renormalization") is necessary if the transformation is to have a fixed point. As for the quasilinear transformations for the Ising systems, we will need to choose the parameter z just right. After computing $H_1(\psi)$ we would rescale the block lattice to make it into a unit lattice.

Just as in the Ising models, this transformation "preserves the partition function".

$$\begin{aligned}\int \exp[-H_1(\psi)]D\psi &= \int \left[\int \prod_\alpha \delta(\psi_\alpha - z\sum_i \phi_{\alpha,i})D\psi\right] \exp[-H(\phi)]D\phi \\ &= \int \exp[-H(\phi)]D\phi\end{aligned}$$

The above transformation is similar to the quasilinear transformations for the Ising systems in that the correlation function of the renormalized system is directly related to the correlation function of the original system. **Work this out.** In particular the correlation length is reduced by the block length. Of course, our definition of the renormalized Hamiltonian H_1 has all the infinite volume problem that we encountered with the Ising systems. The definition of H_1 makes sense if we start with a lattice model in a finite volume. One then needs to show that an infinite volume limit can be taken.

There are lots of variations on the above transformation. For example one can soften the delta function:

$$\exp[-H_1(\psi)] = \int \prod_\alpha c(a) \exp(-a[\psi_\alpha - z\sum_i \phi_{\alpha,i}]^2) \exp[-H(\phi)]D\phi$$

Here a is a parameter and $c(a)$ is the constant defined by

$$\int_{-\infty}^{\infty} c(a) \exp(-ax^2)dx = 1$$

With measures that are defined in terms of Gaussian measures there is another approach to defining renormalization group transformations. This is the approach that we will use, but first we need some more Gaussian process machinery.

7.3 Normal ordering

Next we want to define a procedure called normal ordering or Wick ordering. It will prove useful when we look at the linearization of the renormalization

group map about a Gaussian fixed point. Let $\phi(x)$ be a Gaussian process with probability measure μ. The normal ordered polynomials $:\phi(x)^n:$ are the polynomials one obtains by applying the Gramm-Schmidt orthogonalization procedure to the monomials $\phi(x)^n$. So $:\phi(x)^n:$ is an nth order polynomial in $\phi(x)$ and

$$\int :\phi(x)^n::\phi(x)^m: d\mu = 0$$

if $n \neq m$. The polynomials are normalized so that the coeffecient of $\phi(x)^n$ in $:\phi(x)^n:$ is 1. These polynomials are closely related to Hermite polynomials (they are Hermite polynomials?) The operation of normal ordering is extended to other functions by requiring that it be linear. For example, we define

$$:e^{i\phi(x)}: := \sum_{n=0}^{\infty} \frac{1}{n!} :\phi(x)^n:$$

We claim that

$$:e^{i\phi(x)}: := e^{\frac{1}{2}C(x,x)} e^{i\phi(x)}$$

More generally, we will show

$$:e^{it\phi(x)}: := e^{\frac{1}{2}t^2C(x,x)} e^{it\phi(x)}$$

To prove this we need to show that the t^n term in $e^{\frac{1}{2}t^2C(x,x)}e^{it\phi(x)}$ is $:\phi(x)^n:/n!$. Clearly this term is a polynomial in $\phi(x)$ whose highest order term is $\phi^n(x)$. Thus all we need to show is that the t^n term in $e^{\frac{1}{2}t^2C(x,x)}e^{it\phi(x)}$ is orthogonal to the s^m term in $e^{\frac{1}{2}s^2C(x,x)}e^{is\phi(x)}$. Now a little computation shows

$$\int e^{\frac{1}{2}t^2C(x,x)}e^{it\phi(x)}e^{\frac{1}{2}s^2C(x,x)}e^{is\phi(x)}\, d\mu = \exp[-tsC(x,x)]$$

Clearly the expansion of the right side only contains terms $t^n s^m$ with $n = m$, and so the claim is proven.

The above result also gives a nice way to compute normal ordered monomials. Since

$$:e^{it\phi(x)}: := e^{\frac{1}{2}t^2C(x,x)} e^{it\phi(x)}$$

one can compute the normal ordered polynomials by differentiating with respect to t a bunch of times and then setting $t = 0$. Normal ordering depends on the covariance, but as we can see from the above the dependence

on the covariance is rather simple.

$$: \phi^n(x) := \sum_{k=0}^{n} a_{n,k} C(x,x)^{(n-k)/2} \phi(x)^k$$

where the $a_{n,k}$ are constants which do not depend on the covariance. In fact they are the coeffecients of the Hermite polynomials. $\sum_{k=0}^{n} a_{n,k} x^k$ is the nth Hermite polynomial.

When we implement the renormalization group we will want to integrate out some of the modes in our functional integrals. Gaussian integrals provide a natural way to do this. Suppose that we have a covariance $C(x,y)$ and it may be written as a sum of two covariances

$$C(x,y) = C_1(x,y) + C_2(x,y)$$

If $C_1(x,y)$ is a valid covariance, then there is a Gaussian process $\phi_1(x)$ with this covariance. We denote the underlying measure by μ_1. Likewise, there is a Gaussian process $\phi_2(x)$ with measure μ_2 whose covariance is $C_2(x,y)$. Now consider the product measure $\mu_1 \times \mu_2$. The sums $\phi_1(x) + \phi_2(x)$ are random variables for this measure. Their characteristic function is easily computed

$$\begin{aligned} \int & \exp[i \sum t(x)(\phi_1(x) + \phi_2(x))] d(\mu_1 \times \mu_2) \\ &= \int \exp[i \sum t(x)\phi_1(x)] d\mu_1 \int \exp[i \sum t(x)\phi_2(x)] d\mu_2 \\ &= \exp[-\frac{1}{2} \sum_{x,y} t(x)t(y)C_1(x,y)] \exp[-\frac{1}{2} \sum_{x,y} t(x)t(y)C_2(x,y)] \end{aligned}$$

Thus $\phi_1(x)+\phi_2(x)$ is a Gaussian process with covariance $C_1(x,y)+C_2(x,y) = C(x,y)$.

The operation of normal ordering depends on the underlying measure, or, equivalently the covariance. When we need to make this dependence explicit we will write $: \phi(x)^n :_C$. Normal ordering has a nice property with respect to integrating out part of the covariance.

Lemma: Let $\phi_1(x)$ and $\phi_2(x)$ be Gaussian processes with covariances C_1 and C_2 and measures μ_1 and μ_2. Then letting $C = C_1 + C_2$,

$$\int : (\phi_1(x) + \phi_2(x))^n :_C \, d\mu_2 =: \phi_1(x)^n :_{C_1}$$

Proof:

$$\begin{aligned}
\int :\exp[it(\phi_1(x)+\phi_2(x))]:_C d\mu_2 &= \exp[\frac{1}{2}t^2C(x,x)]\int \exp[it(\phi_1(x)+\phi_2(x))]d\mu_2 \\
&= \exp[\frac{1}{2}t^2C(x,x)+it\phi_1(x)]\int \exp[it\phi_2(x)]d\mu_2 \\
&= \exp[\frac{1}{2}t^2C(x,x)+it\phi_1(x)-\frac{1}{2}t^2C_2(x,x)] \\
&= \exp[\frac{1}{2}t^2C_1(x,x)+it\phi_1(x)] \\
&= :\exp[it\phi_1(x)]:_{C_1}
\end{aligned}$$

The result now follows by expanding both sides in powers of t. □

Suppose that our original system is given by $\exp[-V(\phi)]d\mu(\phi)$ where μ has covariance $C(x,y)$. We split the covariance into a short range and long range part $C(x,y) = C_s(x,y) + C_l(x,y)$. Let $\phi_s(x), \mu_s$ and $\phi_l(x), \mu_l$ be Gaussian processes with covariances C_s and C_l respectively. If we only integrate out ϕ_s, then the result will be a function of ϕ_l.

$$\exp[-V_1(\phi_l)] = \int \exp[-V(\phi_s+\phi_l)]d\mu_s$$

After appropriate rescaling this will be our renormalization group transformation.

There are lots of ways to split the covariance into short and long range parts. The most naive thing to do would be to let $C_s(x,y)$ equal $C(x,y)$ when $|x-y|$ is not large and equal zero otherwise. However, C_s and C_l must both be positive definite, so this splitting cannot be used. When the covariance is diagonal in momentum space, a natural way to do the splitting is in momentum space. Let $\hat{C}(k)$ be the fourier transform of $C(x,y)$. We could let $\hat{C}_s(k)$ equal $C(k)$ when k is large and equal 0 otherwise. The resulting transformation is often called a momentum space RG transformation, and it is these transformations that we are going to study. Note that the transformation is not simply the map $V \to V_1$. Part of the interaction is now encoded in the covariance and the covariance is changing. So we should think of the above as a map from V, C into V_1, C_l. (The rescaling in both V_1 and C_l has yet to be done.)

Exercises:

1. Define : $\exp(t\phi(x))$: by normal ordering its power series term by term. Show that

$$: e^{t\phi(x)} := e^{-\frac{1}{2}t^2 C(x,x)} e^{t\phi(x)}$$

7.4 The Gaussian fixed point

We consider a continuum model with covariance $C(x, y)$ which is given by

$$C(x,y) = \frac{1}{(2\pi)^d} \int \exp[-ik \cdot (x-y)] \hat{C}(k) dk$$

Since we are in the continuum the integral is over all of $\mathbb{R}^d$. We would like to study the covariance $\hat{C}(k) = \frac{1}{k^2+a}$. We will modify it somewhat by assuming that $\hat{C}(k)$ is supported in the unit sphere, i.e., $\hat{C}(k)$ is nonzero only if $|k| \leq 1$. This requires changing $\frac{1}{k^2+a}$ for $|k| > 1$, but such a change should not change the long distance behavior of the covariance. We split this covariance as $C = C_l + C_s$ where $\hat{C}_l(k) = C(k)\chi(|k| \leq \frac{1}{2})$ and $\hat{C}_s(k) = C(k)\chi(|k| > \frac{1}{2})$. So C_l contains the long distance modes and C_s the short distance modes. We let ϕ_l and ϕ_s be Gaussian processes with covariances C_l and C_s.

The renormalization group transformation will be to integrate out ϕ_s and then rescale the field. There are two type of rescaling that must be done. First, we need to rescale the momenta by a factor of 2. This will rescale the sphere $|k| \leq \frac{1}{2}$ back to the unit sphere. This rescaling is the same as rescaling distances by a factor of $\frac{1}{2}$. Second we need to rescale the field ϕ_l by a factor of z, where z is some parameter which will need to be adjusted in order to obtain a fixed point. These two rescalings imply that we should define the "block field" ψ by $\phi_l(x) = z^{-1}\psi(x/2)$. We need to choose the covariance of ψ so that the covariance of $z^{-1}\psi(x/2)$ is $C_l(x, y)$. Let C' be the covariance of ψ. Then the covariance of $z^{-1}\psi(x/2)$ is $z^{-2}C'(x/2, y/2)$. So we want

$$z^{-2}C'(x/2, y/2) = C_l(x, y)$$

or equivalently

$$z^{-2}C'(x, y) = C_l(2x, 2y)$$

and so

$$z^{-2}C'(x,y) = \frac{1}{(2\pi)^d} \int \exp[-ik \cdot (2x-2y)] \hat{C}_l(k) dk$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^d}\int \exp[-ik\cdot(2x-2y)]\chi(|k|\le\frac{1}{2})\hat{C}(k)dk \\
&= 2^{-d}\frac{1}{(2\pi)^d}\int \exp[-ik\cdot(x-y)]\chi(|k|\le 1)\hat{C}(k/2)dk
\end{aligned}
$$

where we did a simply change of variables in the last line. Thus we have

$$\hat{C}'(k) = z^2 2^{-d}\chi(|k|\le 1)\hat{C}(k/2) \tag{2}$$

Our renormalization group transformation is thus given by $C, V \to C', V'$ where C' is defined by the above equation and V' is defined by

$$\exp[-V'(\psi)] = \int \exp[-V(z^{-1}\psi(x/2)+\phi_s(x))]d\mu_s \tag{3}$$

The notation $V(z^{-1}\psi(x/2)+\phi_s(x))$ means that wherever we see a $\phi(x)$ in $V(\phi)$ we replace it by $z^{-1}\psi(x/2)+\phi_s(x)$. For example, if $V(\phi)=\int\phi(x)^4dx$, then $V(z^{-1}\psi(x/2)+\phi_s(x)) = \int[z^{-1}\psi(x/2)+\phi_s(x)]^4dx$.

All of the headaches that we had with the Ising system are still present. There is an infinite volume limit to be taken. The new interaction V' will be much more complicated that the original interaction. In particular, it will not be local, i.e., it will contain terms like $\int\psi(x)\psi(y)k(x,y)dxdy$. There is one advantage to the present setup. Note that $V=0$ does not imply that our system is trivial. A Gaussian measure can have a nonzero correlation length, even an infinite one. Thus a fixed point of the RG which had $V=0$ might be quite interesting. (In the Ising systems $H=0$ is a fixed point, but a very trivial one.) Furthermore, we might hope to study our new RG transformation by perturbing around $V=0$. (There is no analog of this in the Ising systems since any small H will yield a high temperature system.)

To look for a Gaussian fixed point we simply need to see if (2) has a fixed point. (Obviously, if V is zero then V' is also zero). There are two simple fixed points. The first is to simply let $\hat{C}(k)=\chi(|k|\le 1)$ and take $z=2^{d/2}$. If $\hat{C}(k)$ were simply the function 1 then the covariance would be the identity operator. This yields uncoupled fields and a trivial model, in particular an infinite correlation length. Changing the covariance to $\chi(|k|\le 1)$ will only change the short distance behavior. It will still be a trivial model. This fixed point is called the "high temperature" fixed point. The second fixed point is to let

$$
\begin{aligned}
&\hat{C}(k) = k^{-2}\chi(|k|\le 1) \\
&z = 2^{(d-2)/2}
\end{aligned}
$$

Actually, there is nothing special about k^{-2} here. Choosing z appropriately, $|k|^p$ is a fixed point for any power. These fixed points are important if one allows interactions with a power law decay.

For the rest of this section we will take $z = 2^{(d-2)/2}$ so that the transformation has the k^{-2} fixed point. We want to study the linearization of our transformation about the Gaussian fixed point. C' only depends on C, not on V and the map $C \to C'$ is already a linear map. Linearizing $C, V \to V'$ about $V = 0$ is easy:

$$V'(\psi) = \int V(z^{-1}\psi(x/2) + \phi_s(x))d\mu_s$$

We have not said anything about what sort of space the V should belong to and we are not going to now. We will simply look at what this linearization does to local polynomial interactions, i.e., interactions of the form $\int \phi(x)^n dx$. Equivalently, we can use the normal ordered interactions $\int :\phi(x)^n: dx$. The normal ordering here is with respect to the covariance C. The lemma ?? said

$$\int :[\phi_l(x) + \phi_s(x)]^n: d\mu_s =: \phi_l(x)^n :_{C_l}$$

Since $\phi_l(x) = z^{-1}\psi(x/2)$ this implies

$$\int :[\phi(x)]^n: d\mu_s = z^{-n} :\psi(x/2)^n :_{C'}$$

We have used the fact that $: \phi_l(x)^n :_{C_l} = z^{-n} : \psi(x/2)^n :_{C'}$, which follows easily from ??. Thus the linearized map sends $\int :\phi(x)^n : dx$ into $z^{-n} \int : \psi(x/2)^n : dx$ which after a trivial change of variables becomes $z^{-n}2^d \int : \psi(x)^n : dx$. Thus the normal ordered monomials are eigenvectors of the linearized transformation with eigenvalue $z^{-n}2^d$. Since $z = 2^{(d-2)/2}$, the eigenvalues are $2^{d-n(d-2)/2}$.

For the quadratic interaction the eigenvalue is 2^2 for all dimensions. Thus there is always at least one eigenvalue greater than one. If we start with a $V(\phi)$ that is even then V' will be even. So we restrict our attention to the even subspace for the moment. The eigenvalue of the quartic interaction ($n = 4$) is 2^{4-d}. This is less than 1 when $d > 4$. So when $d > 4$ the Gaussian fixed point has only one eigenvalue greater than one in the even subspace. This is the same setup as for the Ising models. The eigenvalue greater than one should be 2^{y_T}, so we find $y_T = 2$. This yields $\nu = 1/y_T = 1/2$, and

$\alpha = 2 - d/y_T = 2 - d/2$. The value for ν is correct. The value for α is negative since $d > 4$, and this is wrong. One can go on and consider odd interactions, find another eigenvalue of the linearization that is greater than one and then compute the critical exponents β, γ and δ. The results are all wrong. The explanation for these wrong values is somewhat involved. We will return to it later. The buzz word is "dangerous irrelevant variables."

So far we have only considered the action of the linearization on local polynomials in the field. Next we consider a particular interaction that is not of this form, namely,

$$\int \nabla\phi(x)^2 dx$$

There are at least two ways to make sense of $\nabla\phi(x)$. First, if the covariance is nice enough, one can prove that there is a version of the Gaussian process for which the random variable is differentiable function of x with probability one. Second, the formal calculation

$$\nabla\phi(x) = \int \delta(x)\nabla\phi(x)dx = -\int \phi(x)\nabla\delta(x)dx$$

suggest that we define $\nabla\phi(x)$ to be $\phi(v)$ where v is the element in the Hilbert space of distributions given by $-\nabla\delta(x)dx$. (Actually this is a triple of elements in the Hilbert space since ∇ is a vector.) The action of the linearized RG is

$$\begin{aligned}
\int\int \quad & [\nabla\phi_l(x) + \nabla\phi_s(x)]^2 dx d\mu_s = \int [\nabla\phi_l(x)]^2 dx + constant \\
= \quad & z^{-2}\int [\nabla\psi(x/2)]^2 dx + constant \\
= \quad & z^{-2}2^d 2^{-2}\int [\nabla\psi(x)]^2 dx + constant
\end{aligned}$$

In the last equality we did the change of variables $x \to 2x$, but obtained an extra factor of 2^{-2} from the gradient. Using our choice of z, $z^{-2}2^d 2^{-2} = 1$. Thus this interaction has eigenvalue 1. This appears to be bad news in view of chapter 2, but this eigenvalue is really an artifact of the continuous spin nature of the model. Recall that we can always do a rescaling of the fields, i.e., send all $\phi(x)$ to $\alpha\phi(x)$ without changing the physics of the model in any significant way. In particular, given a fixed point of the RG we must in fact have a one parameter family of fixed points. We can formally think of the

RG map as a map $H \to H'$ where $H = C^{-1} + V$. Suppose $H(\phi)$ is fixed point so

$$RH(\phi) = H(\phi)$$

Then

$$RH(\alpha\phi) = H(\alpha\phi)$$

Differentiating with respect to α and then setting $\alpha = 1$ we find that the interaction H' is an eigenvector of the linearization with eigenvalue 1 where

$$H'(\phi) = \frac{d}{d\alpha}|_{\alpha=1} H(\alpha\phi)$$

The Gaussian fixed point we have is homogeneous in the field, so one finds that H' is just the RG fixed point $\int (\nabla\phi)^2(x)$. We can remove this eigenvalue of 1 by restrict the space of interactions to a subspace of codimension one by placing a restriction on the interaction that removes the freedom to rescale the fields. For example we could require that the coeffecient of k^2 in the covariance be 1.

We now consider nonlocal interactions, first from a physicist point of view. For simplicity we consider only interactions that are quadratic in the field, e.g.,

$$\int\int \phi(x)\phi(y)k(x-y)dxdy$$

Integrating out the ϕ_s yields

$$\begin{aligned}
\int\int\int \quad & [\phi_l(x) + \phi_s(x)][\phi_l(y) + \phi_s(y)]k(x-y)dxdyd\mu_s \\
= \quad & \int\int \phi_l(x)\phi_l(y)k(x-y)dxdy + constant \\
= \quad & z^{-2}\int\int \psi(x/2)\psi(y/2)k(x-y)dxdy + constant \\
= \quad & z^{-2}2^{2d}\int\int \psi(x)\psi(y)k(2(x-y))dxdy + constant
\end{aligned}$$

We are going to ignore the constant term. If we had used a normal ordered quadratic interaction the constant term would not be there. Pluging in our choice of z the result is of the same form as the interaction we started with except that $k(x)$ is replaced by $2^{2+d}k(2x)$. In momentum space this is the

same as replacing $\hat{k}(p)$ by $2^2\hat{k}(p/2)$. The local quadratic interaction is obtained here by taking $k(x)$ to be a delta function. Then $\hat{k}$ is a constant function and is trivially an eigenvector for the map

$$\hat{k}(p) \to 2^2\hat{k}(p/2)$$

with eigenvalue 2^2 as we saw before. Any monomial in the components of p will be an eigenfunction of this map with eigenvalue $2^2 - m$ where m is the degree of the monomial. If $k(x)$ is rotationally invariant, then $\hat{k}(p)$ will be as well. The lowest order rotationally invariant nonconstant monomial is $|p|^2$. Its eigenvalue is 1. But this interaction is nothing but our old friend $\nabla\phi(x)^2$. All the other rotationally invariant monomials have $m > 2$ and so an eigenvalue less than 1. We have only considered quadratic functions of the field. Higher order monomials in the field can be treated in the same way. From the above we can see that the eigenvalues of the nonlocal interactions will be of the form 2^{-m} times the eigenvalue of the corresponding local interaction. Thus when the local ϕ^4 interaction has eigenvalue less than 1, i.e., when $d > 4$ the nonlocal ϕ^4 interactions will have eigenvalues less than 1.

Monomials in p in momentum space correspond to derivatives of the delta function back in position space. This is not the typical sort of nonlocal interaction that arises in perturbation theoretic implementations of the RG. Interactions of the form $\int\int \phi(x)\phi(y)k(x-y)dxdy$ where $k(x)$ is a nice function with rapid decay are what we will encounter. For a proper treatment of these nonlocal interactions we need to introduce a proper space of interactions, i.e., define a norm, and study the linearized RG map on this space. This has been done rigorously in some models. The only remarks we will make here it that while it is nice to obtain eigenfunctions and eigenvalues as we did in the above highly formal calculation, there is no reason this is will happen when we work in a proper space of interactions. The spectrum make be continuous, etc. However, this is not a problem apriori. All we really want to know is that when we remove a finite number of directions from the linearized map corresponding to the eigenvalues greater than or equal to one, the remainder of the map is a contraction.

We end this section with some of the standard buzz words in renormalization group. Interactions that have an eigenvalue greater than one are said to be "relevant", while those with eigenvalue less than one are said to be "irrelevant." It is also possible to have an eigenvalue equal to one. Such

an interaction is said to be "marginal." For example, ϕ^4 is marginal in four dimensions. $\nabla\phi^2$ is marginal in all dimensions, but this is just a reflection of a trivial symmetry in the model. Nontrivial marginal operators like ϕ^4 in $d = 4$ are interesting since they typically produce logarithmic corrections to the power law divergences of various quantities at the critical point. We will return to this later.

Exercises:

1 (this is a straightforward) The renormalization group transformation of this section split momentum space into $|k| \leq \frac{1}{2}$ and $\frac{1}{2} < |k| \leq 1$. We could just have well split it into $|k| \leq 1/l$ and $1/l < |k| \leq 1$ for any l greater than 1. Repeat the calculations of this section for this transformation. The eigenvalues of the linearization will be different. However, you should still find that the quartic term changes stability at $d = 4$. This transformation rescales lengths by a factor of l, so to find y_T we should write the eigenvalue of the quadratic interaction as l^{y_T}. The y_T you find should not depend on l and so should be the value we found, namely 2.

7.5 A Non-gaussian fixed point

We now turn our attention to $d < 4$. We saw in the last section that in the space of even interactions the Gaussian fixed point has unstable directions in addition to the always unstable ϕ^2 interaction. So we do not expect the Gaussian fixed point to describe the critical behavior anymore and we should look for a new non-Gaussian fixed point. The method for doing this is known as the ϵ-expansion. ϵ is defined to be $4 - d$. The idea is that if $d < 4$ and ϵ is small, then the non-Gaussian fixed point should be close to the Gaussian fixed point. So we can try to compute it by perturbation theory around $V = 0$. Of course, the model is only defined for integer d, so to take ϵ small we will need to extend the definition to noninteger dimension. To the lowest order in ϵ this is easy. The eigenvalues we found depend on d in a way that immediately generalizes to noninteger d. (Mathematically this doesn't make any sense. Given a function defined on the integers there is no unique way to extend in to noninteger values.) At higher orders in ϵ the game is the same although the functions of d that appear are more complicated. For example integrals in momemtum space will produce the surface area of the sphere in d dimensions. There is a formula for this which contains among other

things the gamma function with the dimension entering in the argument. The gamma function is defined for noninteger values.

As we said above, even if we start with a simple interaction like $\int \phi(x)^4$, the new interaction V' will contain all sorts of terms including higher powers of ϕ and nonlocal terms. The nice thing about the ϵ expansion is that to lowest order in ϵ, the fixed point will only contain local quadratic and quartic terms. (This is not obvious apriori.) The RG map to lowest order in ϵ will be a map in a space of only two parameters. The calculations needed are somewhat involved. In the end many of the terms don't matter. We are going to first do the calculation by assuming that various terms can be neglected because they are higher order in ϵ. We will later go back and try to justify this.

We will compute V' by doing perturbation theory around $V = 0$. Formally we obtain this perturbation theory as follows. We introduce a parameter t in front of V and then use (3) to compute V' as a power series in t. $V' = V_1 t + V_2 t^2 + \cdots$.

$$V_n = -\frac{1}{n!}\frac{d^n}{dt^n}|_{t=0} \log \int \exp[-tV(z^{-1}\psi(x/2) + \phi_s(x))]d\mu_s$$

Of course the first term in the expansion just gives the linearization we considered before and it cannot produce a non-Gaussian fixed point by itself. We will need the first two terms in this expansion.

$$V_1 = \int V(z^{-1}\psi(x/2) + \phi_s(x))d\mu_s$$

$$\begin{aligned} V_2 &= -\frac{1}{2}\frac{d^2}{dt^2}|_{t=0} \log \int \exp[-tV(z^{-1}\psi(x/2) + \phi_s(x))]d\mu_s \quad (4) \\ &= \frac{1}{2}\frac{d}{dt}|_{t=0} \frac{\int V(z^{-1}\psi(x/2) + \phi_s(x)) \exp[-tV(z^{-1}\psi(x/2) + \phi_s(x))]d\mu_s}{\int \exp[-tV(z^{-1}\psi(x/2) + \phi_s(x))]d\mu_s} \quad (5) \\ &= -\frac{1}{2}\int V^2(z^{-1}\psi(x/2) + \phi_s(x))d\mu_s + \frac{1}{2}[\int V(z^{-1}\psi(x/2) + \phi_s(x))d\mu_s]^2 \quad (6) \end{aligned}$$

taking the log removes the connected diagrams

As we said before, our computation to lowest order in ϵ will take place in a two dimensional space of interactions — a local quadratic interaction

and a local quartic interaction. We can put the quadratic interaction in the covariance. So we consider a covariance of the form

$$\hat{C}(k) = (k^2 + a)\chi(|k| \leq 1)$$

and a V of the form

$$V = \lambda \int \phi(x)^4 dx$$

We leave it as an easy calculation for the reader to check that the renormalized covariance is

$$\hat{C}'(k) = (k^2 + 4a)\chi(|k| \leq 1)$$

However, this does not mean that the renormalized a is simply $4a$. The renormalized V will contains a quadratic term and we must move this term into the covariance. We will denote the renormalized a and λ by a' and λ'. So far we have

$$\begin{aligned} a' &= 4a + \text{terms to be computed} \\ \lambda' &= \text{terms to be computed} \end{aligned}$$

The linear term $\int V(z^{-1}\psi(x/2) + \phi_s(x))d\mu_s$ is easy to compute. Keeping in mind that the integral of an odd power of ϕ_s vanishes, it is

$$\begin{aligned} & \int V(\phi_l(x) + \phi_s(x))d\mu_s = \lambda \int dx \int [\phi_l(x) + \phi_s(x)]^4 d\mu_s \\ = \;& \lambda \int dx \int [\phi_l(x)^4 + 6\phi_l(x)^2\phi_s(x)^2 + \phi_s(x)^4]d\mu_s \\ = \;& \lambda \int dx[\phi_l(x)^4 + 6\phi_l(x)^2 C_s(x,x) + 3C_s(x,x)^3] \\ = \;& \lambda \int dx[z^{-4}\psi(x/2)^4 + 6z^{-2}\psi(x/2)^2 C_s(x,x) + 3C_s(x,x)^3] \\ = \;& \lambda \int dx[z^{-4}2^d\psi(x)^4 + 6z^{-2}2^d\psi(x)^2 C_s(2x,2x) + 3C_s(x,x)^3] \end{aligned}$$

In the last equation we did a change of variables in the x integral. The last term in the above is just a constant and can be thrown out. The coeffecient of the ψ^4 term is $\lambda z^{-4}2^d = 2^{4-d} = 2^\epsilon$. The coeffecient of the ψ^2 term contains

$C_s(2x, 2x) = C_s(0, 0)$. It is crucial to note that this depends on a. This quantity is given by an integral in momentum space. We define

$$A(a) = C_s(0, 0) = \frac{1}{(2\pi)^d} \int_{\frac{1}{2} \leq |k| \leq 1} \frac{1}{k^2 + a} dk$$

Then the coeffecient of the ψ^2 term is $6z^{-2}2^d A(a)\lambda = 24A(a)\lambda$. We need to move this term into the covariance. The weight for the Gaussian measure contains $-\frac{1}{2}C^{-1}$. There is also a minus sign in front of V' in the exponential, so this term contributes $48A(a)\lambda$ to the renormalized a. We now have

$$\begin{aligned} a' &= 4a + 48A(a)\lambda + \text{terms to be computed} \\ \lambda' &= 2^\epsilon \lambda + \text{terms to be computed} \end{aligned}$$

Next we consider the second order terms in the perturbation theory,i.e., V_2. We must compute

$$-\frac{1}{2} \int V^2(\phi_l(x) + \phi_s(x)) d\mu_s + \frac{1}{2} [\int V(\phi_l(x) + \phi_s(x)) d\mu_s]^2$$

This will contain a variety of terms including terms that are sixth and eighth order in ψ. We will ignore these terms and only compute the second and fourth order terms. Eq. (6) for V_2 contains two terms. The first term involves the integral

$$\int [\phi_l(x) + \phi_s(x)]^4 [\phi_l(y) + \phi_s(y)]^4 d\mu_s$$

When we expand this out, the terms that are fourth order in ϕ_l are

$$\phi_l(y)^4 \int \phi_s(x)^4 d\mu_s + \phi_l(x)^4 \int \phi_s(y)^4 d\mu_s + 36\phi_l(x)^2 \phi_l(y)^2 \int \phi_s(x)^2 \phi_s(y)^2 d\mu_s$$
$$+16\phi_l(x)\phi_l(y)^3 \int \phi_s(x)^3 \phi_s(y) d\mu_s + 16\phi_l(x)^3 \phi_l(y) \int \phi_s(x) \phi_s(y)^3 d\mu_s$$

Now consider the second term in V_2.

$$\int [\phi_l(x) + \phi_s(x)]^4 d\mu_s \int [\phi_l(y) + \phi_s(y)]^4 d\mu_s$$

The terms in this that are fourth order in ψ are

$$\phi_l(y)^4 \int \phi_s(x)^4 d\mu_s + \phi_l(x)^4 \int \phi_s(y)^4 d\mu_s + 36\phi_l(x)^2\phi_l(y)^2 \int \phi_s(x)^2\mu_s \int \phi_s(y)^2 d\mu_s$$
$$+16\phi_l(x)\phi_l(y)^3 \int \phi_s(x)^3 d\mu_s \int \phi_s(y) d\mu_s + 16\phi_l(x)^3\phi_l(y) \int \phi_s(x) d\mu_s \int \phi_s(y)^3 d\mu_s$$

Some of these terms just cancel a term in the previous mess. We find

$$- \quad \frac{1}{2}\int V^2(\phi_l(x)+\phi_s(x))d\mu_s + \frac{1}{2}[\int V(\phi_l(x)+\phi_s(x))d\mu_s]^2$$
$$= -\frac{1}{2}\lambda^2 \int dx \int dy[36\phi_l(x)^2\phi_l(y)^2(\int \phi_s(x)^2\phi_s(y)^2 d\mu_s - \int \phi_s(x)^2 d\mu_s \int \phi_s(y)^2 d\mu_s)$$
$$+32\phi_l(x)\phi_l(y)^3 \int \phi_s(x)^3\phi_s(y) d\mu_s]$$

The last term comes from combining two terms using the symmetry in x and y. The integrals are just integrals of polynomials in ϕ_s which are easily computed.

$$\int \phi_s(x)^2\phi_s(y)^2 d\mu_s - \int \phi_s(x)^2 d\mu_s \int \phi_s(y)^2 d\mu_s = 2C_s(x,y)^2$$
$$\int \phi_s(x)^3\phi_s(y) d\mu_s = 3C_s(x,x)C_s(x,y)$$

Note that both of these answers decay exponentially fast as $|x-y|$ grows. So even though we are obtaining nonlocal interactions, e.g., terms like $\phi_l(x)^2\phi_l(y)^2$ rather than $\phi_l(x)^4$, they are not too badly nonlocal. We now cheat and replace our nonlocal interactions by local ones, i.e.,

$$\begin{aligned} \phi_l(x)^2\phi_l(y)^2 &\quad \to \phi_l(x)^4 \\ \phi_l(x)\phi_l(y)^3 &\quad \to \phi_l(x)^4 \end{aligned}$$

(We will justify this later.) This yields

$$-\frac{1}{2}\lambda^2 \int V^2(\phi_l(x)+\phi_s(x))d\mu_s + \frac{1}{2}[\int V(\phi_l(x)+\phi_s(x))d\mu_s]^2$$
$$= -\frac{1}{2}\int dx \int dy[36\phi_l(x)^4 2C_s(x,y)^2 + 32\phi_l(x)^4 3C_s(x,x)C_s(x,y)]$$

The second term contains $\int dy C_s(x,y) = \hat{C}_s(0) = 0$. For the first term we define

$$B(a) = \int dy C_s(0,y)^2 = \frac{1}{(2\pi)^d} \int \hat{C}_s(k)^2 dk = \frac{1}{(2\pi)^d} \int_{\frac{1}{2}\leq|k|\leq 1} \frac{1}{(k^2+a)^2} dk$$

Then we are left with

$$\begin{aligned} -36B(a)\lambda^2 \qquad & \int dx \phi_l(x)^4 = -36B(a) \int dx z^{-4} \psi(x/2)^4 72\lambda^2 \\ = \; & -36B(a) z^{-4} 2^d \lambda^2 \int dx \psi(x)^4 = -36B(a) 2^\epsilon \lambda^2 \int dx \psi(x)^4 \end{aligned}$$

We now have

$$\begin{aligned} a' &= 4a + 48A(a)\lambda + \text{terms to be computed} \\ \lambda' &= 2^\epsilon \lambda - 36B(a) 2^\epsilon \lambda^2 \end{aligned}$$

So far we have only computed the contribution of the second order perturbation theory to the quartic part of V'. We are neglecting higher powers of ψ, but we still need to compute the quadratic part. It will turn out that this part will not matter to lowest order in ϵ. The contribution to a' will obviously contain a factor of λ^2, so we will merely include a $O(\lambda^2)$ in our equations.

$$\begin{aligned} a' &= 4a + 48A(a)\lambda + O(\lambda^2) \\ \lambda' &= 2^\epsilon \lambda - 36B(a) 2^\epsilon \lambda^2 \end{aligned}$$

This may seem a bit puzzling. We went to a lot of trouble to compute the $O(\lambda^2)$ term in λ', but we are not bothering to compute the corresponding term in a'. The reason will become apparent in a moment when we look for a fixed point of the above equations.

The next step is to look for a fixed point of our renormalization group equations (7). So we want a solution of

$$\begin{aligned} a &= 4a + 48A(a)\lambda + O(\lambda^2) \\ \lambda &= 2^\epsilon \lambda - 36B(a) 2^\epsilon \lambda^2 \end{aligned}$$

Since ϵ is small, $2^\epsilon \approx 1 + \epsilon \log 2$. In the λ^2 term in the second equation we might as well replace 2^ϵ by 1 and $B(a)$ by $B(0)$. Thus the second equation becomes

$$0 = \epsilon\lambda \log 2 - 36B(0)\lambda^2$$

Denoting the fixed point by $a^\star, \lambda^\star$, we have

$$\lambda^\star = \frac{\epsilon \log 2}{36 B(0)}$$

Plugging this into the first equation we have

$$a = 4a + \frac{4}{3}\epsilon \log 2 \frac{A(a)}{B(a)} + O(\lambda^2)$$

The $O(\lambda^2)$ term is $O(\epsilon^2)$ and so contributes to a only at second order in ϵ. We can also take $a = 0$ in the functions $A(a)$ and $B(a)$ since the resulting error in the equation is second order in ϵ. Thus we find

$$a^\star = -\frac{4}{9}\epsilon \log 2 \frac{A(0)}{B(0)}$$

Now we see the crucial difference between the a and λ equations. The linear part of the λ equation is $\lambda' = (1 + \epsilon \log 2)\lambda$. So the total factor in front of the λ is $\epsilon \log 2$ and to find $\lambda^\star$ to first order in ϵ we must include terms of second order in ϵ^2 in this equation. In the a equation the factor in front of a ends up being just 3, so a is determined by the other terms in the equation that are first order in ϵ.

Having found a nontrivial fixed point we compute the linearization of our two parameter RG map about it. This just amounts to computing the Jacobian of (7). We only need to keep terms up to first order in ϵ.

$$\frac{\partial a'}{\partial a} = 4 + 48 A'(0)\lambda$$

$$\frac{\partial a'}{\partial \lambda} = 48 A(0)$$

$$\frac{\partial \lambda'}{\partial a} = 0$$

$$\frac{\partial \lambda'}{\partial \lambda} = 1 + \epsilon \log 2 - 72 B(0)\lambda$$

Thus the two by two matrix for the linearization will be upper triangular (or lower triangular ??) and so the eigenvalues are just the diagonal entries. At the fixed point the diagonal entries are $4 + \frac{4}{3}\epsilon \log 2 \frac{A'(0)}{B(0)}$ and $1 - \epsilon \log 2$. When $\epsilon = 0$ they are 4 and 1 as they should be. For $\epsilon > 0$, the smaller

eigenvalue goes below 1 so our new non-Gaussian fixed point has only one unstable direction. We leave it as an exercise to compute that $\frac{A'(0)}{B(0)} = -1$. Thus the eigenvalue greater than one is $4 - \frac{4}{3}\epsilon \log 2$. Setting this equal to 2^{y_T} we find (to first order in ϵ),

$$y_T = 2 - \frac{1}{3}\epsilon$$

and so

$$\nu = \frac{1}{y_T} = \frac{1}{2} + \frac{1}{12}\epsilon$$

We can compute a second critical exponent using y_T,

$$\alpha = 2 - \frac{d}{y_T} = 2 - \frac{4-\epsilon}{2 - \frac{1}{3}\epsilon} = \frac{1}{6}\epsilon$$

So far we have only considered even interactions. By considering odd interactions as well we can compute y_h to first order in ϵ. We leave this as an exercise for the reader. One finds that $y_h = 3 - \epsilon/2$ and so

$$\begin{aligned} \gamma &= 1 + \frac{1}{6}\epsilon \\ \beta &= \frac{1}{2} - \frac{1}{6}\epsilon \\ \delta &= 3 + \epsilon \end{aligned}$$

In our search for a non-Gaussian fixed point we used the same choice of z ($z = 2^{(d-2)/2}$) then we did for the Gaussian fixed point. This was not justified. We should have expected to need a different rescaling of the field in order to obtain a non-Gaussian fixed point. The fact that this choice worked implies that the correct choice of z is the same as the Gaussian one to first order in ϵ. This implies something about the critical exponent η. Recall that η describes the power law decay of the correlation function at the critical point.

$$< \phi(x)\phi(y) > \sim \frac{c}{|x-y|^{d-2+\eta}}$$

where $<\ >$ denotes expectation with respect to the probability measure $Z^{-1}e^{-V(\phi)}d\mu$. We need to study the behavior of the correlation function $<$

$\phi(x)\phi(y)>$ under the renormalization group transformation We have $\phi(x) = \phi_s(x) + z^{-1}\psi(\frac{x}{2})$ and every measure in sight is an even measure, so

$$< \phi(x)\phi(y) >=< \phi_s(x)\phi_s(y) > +z^{-2} < \psi(\frac{x}{2})\psi(\frac{y}{2}) > \tag{7}$$

The first term should decay exponentiall in $x-y$ since the covariance of $\phi_s(x)$ does. (This is a bit subtle. We need to be sure there is nothing in V which will introduce small momenta.) Now

$$< \psi(\frac{x}{2})\psi(\frac{y}{2}) >= Z^{-1} \int\int \psi(\frac{x}{2})\psi(\frac{y}{2}) \exp[-V(\phi_s(x) + z^{-1}\psi(\frac{x}{2}))]d\mu_s d\mu'$$

where μ' is the measure for the process ψ. (So μ' is μ_l except for some rescaling.) Doing the $d\mu_s$ integral first and keeping in mind the definition of V' we have

$$< \psi(\frac{x}{2})\psi(\frac{y}{2}) >= Z^{-1} \int \psi(\frac{x}{2})\psi(\frac{y}{2}) \exp[-V'(\psi(x))]d\mu' =< \psi(\frac{x}{2})\psi(\frac{y}{2}) >'$$

where $<\ >'$ denotes expectation with respect to the probability measure $z^{-1}e^{-v'}d\mu'$. If we are at a fixed point

$$< \psi(\frac{x}{2})\psi(\frac{y}{2}) >' \sim \frac{c}{|\frac{x}{2} - \frac{y}{2}|^{d-2+\eta}}$$

Thus (7) can hold only if $z^{-2}2^{d-2+\eta} = 1$. Since $z = 2^{(d-2)/2}$, this implies that $\eta = 0$ to first order in ϵ.

We end this section by going back and justifying the various terms that we threw out in our derivation of the renormalization group equations to first order in ϵ. We are not going to prove anything, but will instead give a self consistency argument. We assume that the only terms in the fixed point to first order in ϵ are those that we have already found. We then argue that the neglected terms will not change this fixed point to first order in ϵ. The equation for λ is special because the eigenvalue associated with ϕ^4 is close to 1. The fixed point value of λ is determined by the terms of order $O(\epsilon^2)$ in the equation. We have already computed everything that can contribute from the second order terms in the perturbation theory. Higher order terms in the perturbation theory do not contribute to second order in ϵ. There are however terms from the first order in perturbation theory that might contribute. For example, the terms $\int \phi^6(x)dx$ and $\int \phi^8(x)dx$ could

have $O(\epsilon^2)$ coeffecients and these terms will contribute to λ in first order perturbation theory.

We claim that in fact the $\int \phi^6(x)dx$ term is $O(\epsilon^3)$. The only contribution to $\phi^6(x)$ that is $O(\epsilon^2)$ comes from second order perturbation theory. By calculations similar to the above we find that the only nonzero term is

$$\int dx \int dy \phi_l^3(x)\phi_l^3(y) \int \phi_s(x)\phi_s(y)d\mu_s = \int dx \int dy \phi_l^3(x)\phi_l^3(y)C_s(x,y)$$

The local part of this is $\int dx \phi_l^6(x) \int dy C_s(x,y)$, and we have already seen that

$\int dy C_s(x,y) = 0$. Thus there are no $O(\epsilon^2)$ contributions to the local $\phi^6(x)$ interaction.

The only potential $O(\epsilon^2)$ contribution to the local ϕ^8 interaction is from second order perturbation theory. This contribution is in fact zero since it comes from a nonconnected interaction. **MORE ????????**

The basic idea behind the ϵ expansion is that when ϵ is small the non-Gaussian fixed point is very close to being Gaussian. Thus it may be studied by expansions around Gaussian measures. This strategy applies in a lot of other setting. For example instead of the Ising model we can consider a model in which the spin at each site takes values on the $n-1$ dimensional sphere. This is often called the $O(n)$ model since that is the relevant symmetry group. The field theory analog is to replace $\phi(x)$ by a vector of fields $\phi_1(x), \cdots, \phi_n(x)$. The critical exponents of this model depend on n as well as the dimension d. There is again a Gaussian fixed point and it becomes unstable in less than four dimensions for all n. The non-Gaussian fixed point that describes the critical theory for $d < 4$ is close to being Gaussian when n is large. Thus one can take $d = 3$ and do a "1/n" expansion in lieu of the ϵ expansion.

Another model which has an ϵ expansion without going to noninteger dimensions is the Ising model with a power law interaction. The Hamiltonian is of the form

$$H = \sum_{i,j} \frac{\sigma_i \sigma_j}{|i-j|^{d+p}}$$

where p is a parameter. A slowly decaying interaction is like putting the nearest neighbor model in higher dimensions. If $p < d/2$ then a Gaussian fixed point describes the critical phenomena. For $p > d/2$ a non-Gaussian fixed point takes over. One can then use $\epsilon = p - d/2$ as an expansion

parameter. (Note: the Gaussian fixed point here is not $1/k^2$ but rather $1/k$ to some other power. Exercise ??????)

Exercises:

1 Show that $\frac{A'(0)}{B(0)} = -1$.

2 This is a continuation of exercise 1 from the previous section. Split the momenta into $|k| \leq 1/l$ and $1/l < |k| \leq 1$ where $l > 1$. The calculations in this section used $l = \frac{1}{2}$. Repeat these calculations for general l. In particular compute y_T to first order in ϵ. The answer should be l independent and equal to the result we got in this section.

3 Compute the result of applying the linearized RG transformation to the interaction $\int \phi(x)$. You should find that $y_h = 3 - \epsilon/2$. Show that this leads to the results given for β, δ and γ.