Math 563 - Fall 14 - Homework 1

1. Let E_n be a sequence of events. We define a new event :

$$\{\omega : \exists\ infinite\ I \subset \mathbb{N}\ such\, that\ i \in I \Rightarrow \omega \in E_i\}$$

This event is sometimes written $E_n\, i.o.$, where $i.o.$ stands for "infinitely often."

(a) Show that $E_n\, i.o. = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$

(b) Prove that if $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P(E_n\, i.o.) = 0$. This is sometimes called the "easy half" of the Borel Cantelli lemma.

2. Let X be a simple random variable on a probability space $(\Omega, \mathcal{F}, P)$. (This means that the range of X is finite.) Let $c_1, c_2, \cdots, c_n$ be the values that X takes on. Let $p_j = P(X = c_j)$. Let μ_X be the distribution of X. Give an explicit description of μ_X in terms of the c_j and p_j. (This is not a hard problem.)

3. Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$. This is problem 2.8 in Durrett. For a hint look at problem 2.7 or 2.9.

4. We flip a fair coin infinitely many times. Let X_n be 1 if the nth flip is heads, and 0 if the nth flip is tails. The sample space Ω consists of all sequences of heads and tails. X_n is a real valued function on Ω. In this problem we assume that there is a σ-field $\mathcal{F}$ and a probability measure P such that X_n is a random variable and the probability measure agrees with your intiution. (We will eventually prove such an $\mathcal{F}$ and P exist.) Define

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

Note that $0 \leq X \leq 1$. Find the distribution μ_X of X. Hint: find $P(X \in E)$ when E is an interval of the form $((k-1)/2^n, k/2^n)$ for integers k and n.

5. Let X_n be as in the last problem. Now define

$$Y = \sum_{n=1}^{\infty} \frac{2X_n}{3^n}$$

NB: It is 3^n in the denominator, not 2^n.

(a) Prove that the distribution function F_Y is continuous.

(b) Prove that F_Y is differentiable a.e. with the derivative equal to 0 a.e. Hint: prove that F_Y is constant on the complement of the Cantor set.

(c) Let μ_Y be the distribution of Y. Let m be Lebesgue measure on the real line. Prove that μ_Y and m are mutually singular. This means that there is a Borel set A with $m(A) = 0$ and $\mu_Y(A^c) = 0$.