Math 563 - Fall 14 - Homework 2

1. (from Durrett) Suppose $EY = 0$ and the variance σ^2 of Y is finite. Let $a > 0$. Prove

$$P(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Hint: apply Chebyshev with $\phi(y) = (y+b)^2$ and optimize your result over b.

2. One of the hypotheses of the monotone convergence theorem is that all the random variables X_n are non-negative. Prove that the conclusion of the theorem is true without this hypothesis if we assume that $E|X_1| < \infty$.

3. Let X_n be a sequence of RV's which converges to a random variable X a.s. Let F be a closed subset of $\mathbb{R}$ which does not contain the origin. Prove that

$$\lim_{n\to\infty} P(|X_n - X| \in F) = 0$$

Hint: use continuity of P.

4. (from Durrett) Let $X_1, X_2, \cdots, X_n$ be RV's and suppose that the distribution of $(X_1, X_2, \cdots, X_n)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$ and the Radon-Nikodym derivative is $g_1(x_1)g_2(x_2)\cdots g_n(x_n)$ where the g_i are non-negative measurable functions but need not have integral 1. Prove $X_1, X_2, \cdots, X_n$ are independent.

5. (from Durrett) Let $\Omega = (0,1)$, $\mathcal{F}$ be the Borel sets in $(0,1)$, and let P be Lebesgue measure. Define RV's by

$$X_n(\omega) = \begin{cases} 0 & \text{if } [2^n\omega] \text{ is even} \\ 1 & \text{if } [2^n\omega] \text{ is odd} \end{cases}$$

where $[x]$ is the largest integer less than or equal to x. Prove that $\{X_n\}_{n=1}^{\infty}$ are independent random variables. Note that this gives a rigorous construction of the probability space for flipping a fair coin infinitely many times.

The change of variables theorem says if X is a random variable and f a real-valued measurable function on the real line (with some condition on f), then

$$E[f(X)] = \int_{\mathbb{R}} f(x)\, d\mu_X$$

where μ_X is the distribution of X. The point of the next two problems is to get more explicit formulae that are amenable to computation in the cases that X is discrete or has a density.

6. Let X be a discrete real-valued random variable. Let $x_1, x_2, \cdots$ be its values and let $p_n = P(X = x_n)$. Let g be any real valued function on the real line. Suppose that

$$\sum_n |g(x_n)|\, p_n < \infty$$

Prove that $g(X)$ is a random variable and

$$E[g(X)] = \sum_n g(x_n) p_n$$

Note that I did not say that g was measurable.

7. Let X be a random variable which has a density $f(x)$. This means its distribution function $F(x)$ satisfies $F(x) = \int_{-\infty}^{x} f(u)\, du$, i.e., its distribution is $f(x)$ times Lebesgue measure. Let $g(x)$ be a real-valued measurable function on the real line such that

$$\int_{-\infty}^{\infty} |g(x)|\, f(x)\, dx < \infty$$

Prove that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

where the integrals with respect to dx are integration with respect to Lebesgue measure.

The last three problems are not to be turned in. They are standard problems in a course like 564. I have included them for background.

8. (564 discrete RV's) Let X be a discrete RV, $x_1, x_2, \cdots$ its values and $p_n = P(X = x_n)$. The function $f(x)$ which is p_n at x_n and is 0 at points not equal to one of the x_n is often called the "probability mass function" in 564 level probability courses.

Suppose we have a coin with probability p of heads. (p is not necessarily 1/2.) The following discrete RV's are of interest:

Binomial: Fix a positive integer n. Flip the coin n times and let X be the number of heads. So X can be $0, 1, 2, \cdots, n$.

Geometric: Flip the coin until you get heads for the first time. Let X be the number of tails. (Warning: depending on the author the definition of X

is sometimes taken to be the total number of flips, including the final one that gave heads.) So the possible values of X are the nonnegative integers.
Negative binomial: Fix an integer r. Flip the coin until we get heads for the rth time. Let X be the total number of tails. So the possible values of X are the nonnegative integers.

For each of these find the probability mass function and the expected value of X.

9. (564 continuous RV's) In a 564 level course, "continuous RV" usually means that the distribution of the RV is absolutely continuous with respect to Lebesgue measure and so is given by $f(x)dx$. (Having a density implies the distribution function is continuous, but the converse is very false.) Here are three common "continuous" RV's.
Uniform: We say X is uniform on $[a, b]$ if $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and $f(x) = 0$ otherwise.
Exponential: We say X has an exponential distribution if $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Here λ is a positive parameter.
Normal: We say X has a normal distribution if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Here μ is a real parameter and σ^2 is a positive parameter.

Find the expected value of each of these random variables.

10. Let X be a random variable with a normal distribution. Find the density function of X^2. Hint: first find an expression for the distribution function of X^2.