Math 563 - Fall '14 - Homework 3

1. Let $\Omega = \{a, b, c, d\}$. Let $\mathcal{F}$ be the σ-field containing all subsets of Ω. Let $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{F}$ but there exist two different probability measures on $(\Omega, \mathcal{F})$ which agree on $\mathcal{E}$.

2. (from Durrett)
(a) Let X and Y be independent random variables which take values in the integers. Prove that the distribution of $X + Y$ is given by

$$P(X + Y = n) = \sum_{m=-\infty}^{\infty} P(X = m)P(Y = n - m)$$

(b) X has a Poisson distribution with parameter $\lambda > 0$ if it takes on the values $0, 1, 2, \cdots$ and

$$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

Show that if X and Y are independent random variables, X has a Poisson distribution with parameter λ and Y has a Poisson distribution with parameter μ, then $X + Y$ has a Poisson distribution. What is the parameter for $X + Y$?

3. Let $X_1, \cdots, X_n$ be random variables and let $Y_1, \cdots, Y_n$ be random variables. Suppose that their joint distribution functions are equal:

$$F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = F_{Y_1,\cdots,Y_n}(x_1, \cdots, x_n)$$

Use the $\pi - \lambda$ theorem to prove that their joint distributions are equal, i.e.,

$$\mu_{X_1,\cdots,X_n} = \mu_{Y_1,\cdots,Y_n}$$

where the two measures are Borel measures on $\mathbb{R}^n$. There is a theorem that says a Borel measure on $\mathbb{R}^n$ is determined by its values on the rectangles. You should not use this theorem.

4. Suppose X_n are identically distributed, uncorrelated, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

(b) Now suppose that you also know that $EX_n^4 < \infty$ and the X_n are independent. Prove that $P(|\frac{1}{n}\sum_{k=1}^n X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p.

5. (from Kallenburg)

(a) Let X_n be an independent sequence of random variables which take values in $[0,1]$. Prove that

$$E[\prod_{n=1}^{\infty} X_n] = \prod_{n=1}^{\infty} EX_n$$

(b) Let A_n be an independent sequence of events Prove that

$$P(\cap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n)$$