Math 563 - Fall '14 - Homework 4

1. (from Durrett) *Monte Carlo integration:* Let $f(x)$ be a Lebesgue integrable function on $[0,1]$. Let U_n be an i.i.d. sequence with each U_n uniformly distributed on $[0,1]$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} f(U_k) \to \int_0^1 f(x)\,dx$$

with probability one. This is a really bad way to compute a one-dimensional integral, but there are analogs of this in higher dimension that are sometimes one of the best ways to compute high dimensional integrals.

2. (from Durrett) *metric for convergence in probability:* We look at the set of all random variables and consider two random variables to be the same if there are equal a.s. Define

$$d(X,Y) = E\left[\frac{|X-Y|}{|X-Y|+1}\right]$$

(a) It is obvious that d is reflexive and $d(X,Y)=0$ if and only if $X=Y$ a.s. Prove that d satisfies the triangle inequality and so defines a metric.
(b) Prove that $X_n \to X$ in probability if and only if $d(X_n,X) \to 0$.

3. (from Durrett) Prove that the metric space in the previous problem is complete.

4. (from Durrett) Let X_n be an independent sequence of RV's. Prove that $sup_n X_n < \infty$ a.s. if and only if there is a constant C such that

$$\sum_{n=1}^{\infty} P(X_n > C) < \infty.$$

5. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. (We also assume $\{X_n, Y_n : n = 1,2,3,\cdots\}$ is independent.) Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0,t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1]+E[Y_1]} \quad a.s.$$

6. (from Durrett) Let X_n be an i.i.d. sequence of random variables with distribution function $F(x)$. Let λ_n be an increasing sequence of real numbers which converges to ∞. Let

$$A_n = \{\max_{1 \le m \le n} X_m > \lambda_n\}$$

Prove that if $\sum_n (1 - F(\lambda_n)) < \infty$, then $P(A_n \ i.o.) = 0$, and that if the sum is ∞ then the probability is 1.