Math 563 - Fall '14 - Homework 5

Do 7 of the 8 problems

1. A random variable has an exponential distribution with parameter λ if the distribution is absolutely continuous with respect to Lebesgue measure and the density is $f(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. The mean is easily found to be $1/\lambda$. Let X_n be an i.i.d. sequence with exponential distribution. For $x > 1/\lambda$ find

$$\lim_{n\to\infty} \frac{1}{n} \ln(P(\overline{X_n} \geq x))$$

As always $\overline{X_n} = \frac{1}{n}\sum_{k=1}^{n} X_k$. You should use the big theorem we proved for large deviations, so this problem is essentially computational.

2. (from Durrett, Fatou's lemma) Suppose that $X_n \Rightarrow X$ and $g \geq 0$ is a continuous real valued function on the reals. Prove

$$\lim_{n\to\infty} E[g(X_n)] \geq E[g(X)]$$

3. (from Durrett) Let X_n be a sequence of integer valued random variables, X another integer valued random variable. Prove that X_n converge to X in distribution if and only if

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

for all integers m.

4. Take a coin with the probability of heads equal to $p \in (0,1)$ and flip it n times. Let X be the number of heads you get. The distribution of X is called the binomial distribution with n trials. It is easy to show that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots n$$

Now fix a $\lambda > 0$, and let X_n have a binomial distribution with n trials and $p = \lambda/n$. Prove that $X_n \Rightarrow X$ where X has the Poisson distribution with parameter λ, i.e.,

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

5. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to the random variable c. Prove or disprove each of the following
(a) X_n converges to c in probability
(b) X_n converges to c a.s.

6. Let μ_n be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where $f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that μ_n converges in distribution to μ where μ is $f(x)$ time Lebesgue measure.

7. (from Durrett, converging together lemma) Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where c is a constant. Prove that $X_n + Y_n \Rightarrow X + c$. Note that this implies that if $X_n \Rightarrow X$ and $Y_n - X_n \Rightarrow 0$, then $Y_n \Rightarrow X$.

8. (from Durrett, the Levy metric) For distribution functions F and G define

$$d(F, G) = \inf\{\epsilon > 0 : F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon\}$$

Prove that d is a metric (don't forget to show it is reflexive) and that F_n converges to F in distribution if and only if $d(F_n, F) \to 0$.