Math 563 - Fall '14 - Homework 6

Do 6 of the 7 problems

1. The Cauchy distribution has the density function

$$f(x) = \frac{1}{\pi(1+x^2)}$$

I.e., the distribution is Lebesgue measure on the real line times this function.
(a) Compute the characteristic function of a Cauchy random variable. Hint: contour integration.
(b) Show that if X_n is an i.i.d. sequence with the Cauchy distribution, then for all n, $\frac{1}{n}\sum_{k=1}^{n} X_k$ has the Cauchy distribution.

2. Suppose that X_n converges in distribution to X and Y_n converges in distribution to Y. Suppose further that for each n, X_n and Y_n are independent, and that X and Y are independent. Assume that X_n, Y_n, X, Y are defined on the same probability space. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

3. (from Durrett) Let X_n be an iid sequence such that for all n, $X_n \geq 0$, $EX_n = 1$, $var(X_n) = \sigma^2$ where $\sigma \in (0, \infty)$. Let $S_n = \sum_{k=1}^{n} X_k$. Prove that $\sqrt{S_n} - \sqrt{n}$ converges in distribution to a normal distribution and find the mean and variance of the limiting normal distribution.

4. Let E_n be independent events such that $\sum_n P(E_n) = \infty$ and $P(E_n) \to 0$. Prove there are increasing sequences c_n and d_n, both of which go to ∞ such that

$$\frac{\sum_{k=1}^{n} 1_{E_k} - c_n}{d_n} \Rightarrow Z$$

where Z is a random variable with the standard normal distribution.

5. Let X be a real valued random variable. Let $\phi(t)$ be its characteristic function.

(a) Show that if there is an $a \neq 0$ such that $\phi(2\pi a) = 1$, then aX takes values in the integers a.s.
(b) Show that if there is an $a \neq 0$ such that $|\phi(2\pi a)| = 1$, then there is a real constant b such that $aX + b$ takes values in the integers a.s.
(c) Show that if there is an interval (a, b) such that $|\phi(t)| = 1$ for $t \in (a, b)$, then X is a constant a.s.

6. Let X_n be an i.i.d. sequence with $EX_n = \mu$ and $Var(X_n) = \sigma^2 < \infty$. The sample mean is defined by

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

By the strong law of large numbers it converges a.s. to μ. The sample variance is defined by

$$V_n = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2$$

(a) Prove that V_n converges a.s. to σ^2. Hint: the variance is defined as $E[(X-\mu)^2]$. It is also equal to $E[X^2] - \mu^2$. The sample variance is almost equal to

$$\frac{1}{n} \sum_{k=1}^{n} X_k^2 - \overline{X}_n^2$$

(b) (loosely based on exercise 4.5, "self-normlized sums", in Durrett) Prove that

$$\frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{nV_n}}$$

converges in distribution to the standard normal distribution.

7. The gamma distribution is a two parameter family of densities. The parameters λ and w are both positive. The density is

$$f(x) = \frac{\lambda^w}{\Gamma(w)} x^{w-1} \exp(-\lambda x)$$

for $x \geq 0$. The density is 0 for $x < 0$. The gamma function is defined by

$$\Gamma(w) = \int_0^\infty x^{w-1} e^{-x} \, dx$$

Integration by parts shows $\Gamma(w+1) = w\Gamma(w)$. If w is an integer then $\Gamma(w) = (w-1)!$. A little calculus shows the mean of the gamma distribution is w/λ and the variance is w/λ^2. For positive integers n let X_n be a random variable with a gamma distribution with $w = n$ and $\lambda = 1$. Prove that $(X_n - n)/\sqrt{n}$ converges in distribution to a standard normal.