Math 563 - Fall '14 - Homework 7

Do 6 of the 7 problems

1. (from Durrett) Thinning a process distribution. Let N have a Poisson distribution with parameter λ. (Note N is not a Poission process, just a single RV.) Let X_n be an i.i.d. sequence taking only the values 0 and 1. Assume also that N and $\{X_n\}_{n=1}^{\infty}$ are independent. Let

$$N_0 = |\{n \leq N : X_n = 0\}|, \qquad N_1 = |\{n \leq N : X_n = 1\}|$$

Here $|\quad|$ just means the cardinality of the set. Prove that N_0 and N_1 are independent, and each is Poisson with parameters $(1-p)\lambda$ and $p\lambda$ where $p = P(X_n = 1)$.

2. (from Durrett) Let $\Omega = \{a, b, c\}$. Given an example of a random variable X and σ-fields $\mathcal{F}_1$,$\mathcal{F}_2$ such that

$$E[E[X|\mathcal{F}_1]|\mathcal{F}_2] \neq E[E[X|\mathcal{F}_2]|\mathcal{F}_1]$$

3. (from Durrett) Define $var[X|\mathcal{F}]$ to be $E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$. Prove that

$$var(X) = E[var[X|\mathcal{F}]] + var(E[X|\mathcal{F}])$$

(This is short).

4. (from Durrett) Let Y_n be an i.i.d. sequence of random variables. Let N be a random variable taking values in the positive integers which is independent of $\{Y_n\}_{n=1}^{\infty}$. Let μ and σ^2 be the mean and variance of the Y_n. Let

$$X = \sum_{n=1}^{N} Y_n.$$

So the number of terms in the sum is random. Show that

$$var(X) = \sigma^2 E[N] + \mu^2 var(N)$$

5. Let X_n be an independent sequence of random variables. Define a power series by

$$f(z) = \sum_{n=1}^{\infty} X_n z^n$$

The radius of convergence of this power series depends on the X_n, so it is a random variable. Prove that this radius is equal to a constant a.s.

6. (from Kallenburg) Let X, Y, Z, W be random variables such that (X, Y) has the same distribution as (Z, W). X is integrable. Prove that $E[X|Y]$ has the same distribution as $E[Z|W]$. The notation $E[X|Y]$ means $E[X|\sigma(Y)]$. Hint: recall problem 3 from the first homework which showed that any $\sigma(Y)$ measurable function can be written in the form $f(Y)$, where f is a Borel-measurable function on the real line. So $E[X|Y]$ can be written in the form $f(Y)$. Show that $E[Z|W]$ is $f(W)$.

7. Let X be a random variable with $E|X| < \infty$. In general the σ-field generated by $|X|$ will be smaller than the σ-field generated by X. The goal of this problem is to find an explicit formula for $E[X|\sigma(|X|)]$ in a couple of special cases.
(a) Suppose X is a discrete RV (countable range). Find an explicit formula for $E[X|\sigma(|X|)]$.
(b) Now suppose the distribution of X is absolutely continuous with respect to Lebesgue measure and the density function is $f(x)$. Use your answer to part (a) to guess a formula for $E[X|\sigma(|X|)]$, and then show that your guess is correct. Your formula should involve f.