**Math 563 - Fall 15 - Homework 1**

1. Let $E_n$ be a sequence of events. Let

$$A = \limsup_{n\to\infty} E_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$$
$$B = \liminf_{n\to\infty} E_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k$$

Prove that

$$1_A = \limsup_{n\to\infty} 1_{E_n}$$
$$1_B = \liminf_{n\to\infty} 1_{E_n}$$

2. Let $E_n$ be a sequence of events. Recall that $\limsup_{n\to\infty} E_n$ is defined to be $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. This event can be thought of as the event that $E_n$ occurs infinitely often. Prove that if $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P(E_n \, i.o.) = 0$. This is sometimes called the "easy half" of the Borel Cantelli lemma.

3. Let $E_n$ be a sequence of events with $P(E_n) = 1$ for all $n$. Prove that

$$P(\cap_{n=1}^{\infty} E_n) = 1$$

4. (from Resnick) Let $\mathcal{F}$ be a $\sigma$-field in $\Omega$. Suppose $Q$ is a function from $\mathcal{F}$ into $[0,1]$ such that $Q(\Omega) = 1$, $Q$ is finitely additive on $\mathcal{F}$, and $Q$ has the following property. If $E_n \in \mathcal{F}$ are pairwise disjoint and their union is $\Omega$, then $\sum_{n=1}^{\infty} Q(E_n) = 1$. Prove that $Q$ is a probability measure.

5. (from Resnick) Let $P_1, P_2$ be two probability measures on $(\Omega, \mathcal{F})$. You might think that if they agree on a collection $\mathcal{C}$ of events, and the $\sigma$-field generated by $\mathcal{C}$ is $\mathcal{F}$, then the two probability measures agree on all of $\mathcal{F}$. This is not true. Show that the following gives a counterexample.

$\Omega = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$

$$P_1(\{a\}) = P_1(\{d\}) = P_2(\{b\}) = P_2(\{c\}) = \frac{1}{6},$$
$$P_1(\{b\}) = P_1(\{c\}) = P_2(\{a\}) = P_2(\{d\}) = \frac{1}{3}$$

6. (from Resnick) Let $P$ be a probability measure on $\mathcal{B}(\mathbb{R})$, the Borel sets in $\mathbb{R}$. Prove that for any $E \in \mathcal{B}(\mathbb{R})$ and any $\epsilon > 0$ there exits a finite union of disjoint intervals $A$ such that $P(E \Delta A) < \epsilon$.

Hint: Define $\mathcal{F}$ to be the collection of Borel sets such that for all $\epsilon > 0$ there exits a finite union of disjoint intervals $A$ such that $P(E \Delta A) < \epsilon$. What can you prove about $\mathcal{F}$?