Math 563 - Fall 15 - Homework 1

6. (from Resnick) Let P be a probability measure on $\mathcal{B}(\mathbb{R})$, the Borel sets in $\mathbb{R}$. Prove that for any $E \in \mathcal{B}(\mathbb{R})$ and any $\epsilon > 0$ there exits a finite union of disjoint intervals A such that $P(E\Delta A) < \epsilon$.

Hint: Define $\mathcal{F}$ to be the collection of Borel sets such that for all $\epsilon > 0$ there exits a finite union of disjoint intervals A such that $P(E\Delta A) < \epsilon$. What can you prove about $\mathcal{F}$?

Solution: I am interpretting "intervals" to mean all intervals that are open, closed or half-open/half-closed, along with singletons, the entire real line and half-infinite intervals.

Clearly $\mathcal{F}$ contains all sets which are intervals. The σ-algebra generated by the intervals is the Borel sets. So if we can show that $\mathcal{F}$ is a σ-algebra, then $\mathcal{F}$ must be all the Borel sets.

Closed under complements: Let $E \in \mathcal{F}$. Let $\epsilon > 0$. Then there is a set A which is the disjoint union of a finite collection of intervals such that $P(A\Delta E) < \epsilon$. The set A^c is a finite disjoint union of intervals. And it is easy to check that $A\Delta E = A^c\Delta E^c$. So $P(A^c\Delta E^c) < \epsilon$. Thus $E^c \in \mathcal{F}$.

Closed under finite unions: By induction it is enough to show it is closed under the union of two sets. So let $E_1, E_2 \in \mathcal{F}$. Let $\epsilon > 0$. Then there are sets A_1, A_2 which are disjoint unions of intervals such that $P(E_i\Delta A_i) < \epsilon/2$ for $i = 1, 2$. Let $A = A_1 \cup A_2$. Note that A is a finite disjoint union of intervals. It is easy to check that

$$A\Delta(E_1 \cup E_2) \subset (A_1\Delta E_1) \cup (A_2\Delta E_2)$$

So

$$P(A\Delta(E_1\cup E_2)) \leq P((A_1\Delta E_1)\cup(A_2\Delta E_2)) \leq P(A_1\Delta E_1)+P(A_2\Delta E_2) < \epsilon/2+\epsilon/2$$

Thus $E_1 \cup E_2 \in \mathcal{F}$.

Closed under countable unions: We have shown it is a field, so by the usual trick of writing a union as a disjoint union, it is enough to show it is closed under disjoint countable unions. So let $E_n \in \mathcal{F}$ be disjoint and let $E = \cup_{n=1}^{\infty} E_n$. Let $\epsilon > 0$. Since they are disjoint,

$$\sum_{n=1}^{\infty} P(E_n) \leq 1$$

So we can pick N so that

$$\sum_{n=N+1}^{\infty} P(E_n) < \epsilon/2$$

We have $\cup_{n=1}^{N} E_n \in \mathcal{F}$. So there is a set A which is a finite disjoint union of intervals such that $P((\cup_{n=1}^{N} E_n)\Delta A) < \epsilon/2$. We have

$$E\Delta A \subset ((\cup_{n=1}^{N} E_n)\Delta A) \cup (\cup_{n=N+1}^{\infty} E_n)$$

So

$$P(E\Delta A) \leq P((\cup_{n=1}^{N} E_n)\Delta A) + P(\cup_{n=N+1}^{\infty} E_n) < \epsilon/2 + \epsilon/2$$

Thus $E \in \mathcal{F}$.