Math 563 - Fall 15 - Homework 2

1. This problem is an introduction to the Kolmogorov extension theorem which we will prove later. Let Ω be the set of real valued sequences. So an element of Ω looks like $\omega = (\omega_1, \omega_2, \omega_3, \cdots)$ where the ω_i are real numbers. A subset E is called a finite dimensional rectangle if there is a positive integer n and $-\infty \le a_i \le b_i \le \infty$ for $i = 1, 2, \cdots, n$ such that
$E = \{\omega : \omega_i \in (a_i, b_i] \, for \, i = 1, 2, \cdots, n\}$.
(a) Show that the collection of finite dimensional rectangles is a semi-algebra. (not hard)
(b) For $n = 1, 2, 3, \cdots$, let μ_n be a probability measure on the Borel sets in $\mathbb{R}^n$. We say they are *consistent* if for all n and $-\infty \le a_i < b_i \le \infty$ for $i = 1, 2, \cdots, n$, we have

$$\mu_{n+1}((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n] \times \mathbb{R}) = \mu_n((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n])$$

Let $\mathcal{F}$ be the σ-field in Ω generated by the finite dimensional rectangles. The Kolmogorov extension theorem says there is a unique probability measure P on $(\Omega, \mathcal{F})$ such that

$$P(\{\omega : \omega_i \in (a_i, b_i] \, for \, i = 1, 2, \cdots, n\}) = \mu_n((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n])$$

Prove the uniqueness part of the theorem, i.e., if two probability measures on $(\Omega, \mathcal{F})$ both satisfy the above then they are equal on $\mathcal{F}$.

2. (from Resnick) A *countable partition* of Ω is a countable disjoint collection of sets whose union is all of Ω. Let $\{E_n\}_{n=1}^{\infty}$ be such a partition and let $\mathcal{F}$ be the σ-field generated by the collection sets $\{E_n\}_{n=1}^{\infty}$.
(a) Given an explicit description of the sets in $\mathcal{F}$.
(b) Let $X : \Omega \to \mathbb{R}$. Prove that X is measurable with respect to $\mathcal{F}$ if and only if there are constants c_n such that

$$X = \sum_{n=1}^{\infty} c_n 1_{E_n}$$

Notation comment: If X is a random variable, then the distribution of X is a probability measure on the Borel sets in $\mathbb{R}$ which I will denote μ_X. It is defined by $\mu_X(B) = P(X \in B)$ where B is a Borel set. Resnick denotes it by $P \circ X^{-1}$.

3. Let X be a simple random variable on a probability space $(\Omega, \mathcal{F}, P)$. (This means that the range of X is finite.) Let $c_1, c_2, \cdots, c_n$ be the values that X

takes on. Let $p_j = P(X = c_j)$. Let μ_X be the distribution of X. Give an explicit description of μ_X in terms of the c_j and p_j. (This is not a hard problem, and your answer can be quite short.)

4. (from Durrett) Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

Hint: First reduce to the case that $Y \geq 0$. Show that $\{\omega : m2^{-n} \leq Y(\omega) < (m+1)2^{-n}\} = X^{-1}(B_{n,m})$, where $B_{n,m}$ is a Borel set in $\mathbb{R}$. Now let $f_n(x) = m2^{-n}$ for $x \in B_{n,m}$. Show that $f_n(x)$ converges pointwise and let $f(x)$ be the limit.

5. We flip a fair coin infinitely many times. Let X_n be 1 if the nth flip is heads, and 0 if the nth flip is tails. The sample space Ω consists of all sequences of heads and tails. X_n is a real valued function on Ω. In this problem we assume that there is a σ-field $\mathcal{F}$ and a probability measure P such that X_n is a random variable and the probability measure agrees with your intiution. (We will eventually prove such an $\mathcal{F}$ and P exist.) Define

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

Note that $0 \leq X \leq 1$. Find the distribution μ_X of X. Hint: find $P(X \in E)$ when E is an interval of the form $((k-1)/2^n, k/2^n)$ for integers k and n.

6. **(Optional: I will not grade this one.)** Let X_n be as in the last problem. Now define

$$Y = \sum_{n=1}^{\infty} \frac{2X_n}{3^n}$$

NB: It is 3^n in the denominator, not 2^n.

(a) Prove that the distribution function F_Y is continuous.

(b) Prove that F_Y is differentiable a.e. with the derivative equal to 0 a.e. Hint: prove that F_Y is constant on the complement of the Cantor set.

(c) Let μ_Y be the distribution of Y. Let m be Lebesgue measure on the real line. Prove that μ_Y and m are mutually singular. This means that there is a Borel set A with $m(A) = 0$ and $\mu_Y(A^c) = 0$.