Math 563 - Fall 15 - Homework 2 - Solution

4. (from Durrett) Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

Solution: If $Y = f(X)$ for a Borel measurable f, then Y is the composition of measurable functions and so is measurable.

Now suppose Y is $\sigma(X)$ measurable. The positive and negative parts of Y are then both $\sigma(X)$ measurable. So if they can be written as $Y^+ = f_1(X)$ and $Y^- = f_2(X)$ for Borel measurable f_1, f_2, then we can set $f = f_1 - f_2$ and have $Y = f(X)$.

So now we assume $Y \geq 0$ and follow the hint. For a positive integer n and any integer m, the event $\{\omega : m2^{-n} \leq Y(\omega) < (m+1)2^{-n}\}$ is in $\sigma(X)$. So it can be written as $X^{-1}(B_{n,m})$ for a Borel set $B_{n,m}$ in $\mathbb{R}$. Note that $B_{n,m}$ and $B_{n,k}$ need not be disjoint since there can be real numbers that are not in the range of Y but are in both of these Borel sets. So if we just define $f_n(x) = m2^{-n}$ for $x \in B_{n,m}$, this is not well defined. Instead we define

$$g_n(x) = \sum_{m=0}^{m} m2^{-n}\, 1_{B_{n,m}}$$

This sum might be ∞, but this will still be measurable with respect to the Borel sets in the extended reals. Now let $f_n(x) = g_n(x)$ when $g_n(x)$ is finite and $f_n(x) = 0$ when $g_n(x)$ is infinite. Then f_n is Borel measurable.

Now let $\omega \in \Omega$. Then for each n there is a unique m_n such that $m_n 2^{-n} \leq Y(\omega) < (m_n+1)2^{-n}$. We have $\omega \in X^{-1}(B_{n,m_n})$, so $X(\omega) \in B_{n,m_n}$. And ω cannot be in any other $B_{n,k}$ for $k \neq m_n$. So for $x = X(\omega)$, the sum defining $g_n(x)$ has only one nonzero term. We have $f_n(X(\omega)) = m_n 2^{-n}$ and so $|Y(\omega) - f_n(X(\omega))| < 2^{-n}$. Thus $f_n(X(\omega))$ converges to $Y(\omega)$. This shows f_n converges on the range of X, but off the range we don't know it converges. The set where it does converge is Borel measurable, so we can define f to be $\lim_n f_n$ when the limits exits and to be 0 when the limit does not exist. Then f is Borel measurable and $f(X(\omega)) = Y(\omega)$.