Math 563 - Fall 15 - Homework 3

1. (from Resnick, slightly modified) Let X_n be a sequence of RV's on $(\Omega, \mathcal{F}, P)$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

Let $\tau = \inf\{n > 0 : S_n > 0\}$. Note that τ may sometimes be ∞.
(a) Prove τ is a random variable, i.e., it is measurable. Note that τ takes values in the extended reals, so by measurable I mean measurable with respect to the Borel sets in the extended reals.
(b) Define X to be S_τ when τ is finite and 0 when $\tau = \infty$. Prove X is a random variable.

2. Let A_n be independent events. Prove that

$$P(\cap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n)$$

3. (from Durrett) Let $\Omega = (0, 1)$, $\mathcal{F}$ be the Borel sets in $(0, 1)$, and let P be Lebesgue measure. Define RV's by

$$X_n(\omega) = \begin{cases} 0 & \text{if } [2^n\omega] \text{ is even} \\ 1 & \text{if } [2^n\omega] \text{ is odd} \end{cases}$$

where $[x]$ is the largest integer less than or equal to x. Prove that $\{X_n\}_{n=1}^{\infty}$ are independent random variables. Note that this gives a rigorous construction of the probability space for flipping a fair coin infinitely many times.

4. Prove that a random variable X is independent of itself if and only if there is a constant c such that $P(X = c) = 1$. Hint: What can you say about the distribution function of X?
Optional challenge: Prove that a random element $X : \Omega \to \Omega'$ is independent of itself if and only if there is a $c \in \Omega'$ such that $P(X = c) = 1$. Warning: I don't know how to do this.

5. (from Resnick) Let A_n be a sequence of independent events with $P(A_n) < 1$ for all n. Prove

$$P(\cup_{n=1}^{\infty} A_n) = 1 \, if \, and \, only \, if \, P(A_n \, i.o.) = 1$$

Give a counterexample to show that the condition $P(A_n) < 1$ cannot be dropped.

6. (from Resnick) Let X_n be a sequence of independent random variables. Prove

$$P(\sup_n X_n < \infty) = 1$$

if and only if there is a finite constant M such that

$$\sum_{n=1}^{\infty} P(X_n > M) < \infty$$