Math 563 - Fall 15 - Homework 3 - selected solutions

4. Prove that a random variable X is independent of itself if and only if there is a constant c such that $P(X = c) = 1$. Hint: What can you say about the distribution function of X?

Solution Consider the event $X \leq a$ for a real number a. It must be independent of itself, so it can only have probability 0 or 1. So the distribution function $F(x)$ only takes on the values 0 and 1. Consider $\{x : F(x) = 0\}$. Since $F(x)$ converges to 0 at $-\infty$, this set cannot be empty and since it conveges to 1 at ∞, this set is bounded above. So if we let $c = \sup\{x : F(x) = 0\}$, c will be a finite number. By def of the sup, there are x_n with $F(x_n) = 0$ and x_n increasing to c. Letting $F(c-)$ be the left hand limit of F at c, this shows $F(c-) = 0$. This implies $P(X < c) = 0$. Also by def of the sup, $F(c+1/n)$ is not 0 and so must be 1. Since F is right continuous, $F(c) = 1$. So $P(X \leq c) = 1$. Thus $P(X = c) = P(X \leq c) - P(X < c) = 1$.

6. (from Resnick) Let X_n be a sequence of independent random variables. Prove

$$P(\sup_n X_n < \infty) = 1$$

if and only if there is a finite constant M such that

$$\sum_{n=1}^{\infty} P(X_n > M) < \infty$$

Solution $\Rightarrow$: We prove the contrapositive. So assume for all constants M we have

$$\sum_{n=1}^{\infty} P(X_n > M) = \infty$$

We must show this implies $P(\sup_n X_n < \infty) < 1$. By Borel-Cantelli, we know that for all M, $P(X_n > M i.o.) = 1$. If $X_n > M$ infinitely often, then we certainly have $\sup_n X_n > M$. So for all M we have that $P(\sup_n X_n > M) = 1$. Now we restrict to M which are postive integers. The countable intersection of events with probability 1 has probability 1, so $P(\cap_{m=1}^{\infty}\{\sup_n X_n > m\}) = 1$. But the event $\cap_{m=1}^{\infty}\{\sup_n X_n > m\}$ is the event $\sup_n X_n = \infty$. So $P(\sup_n X_n = \infty) = 1$. So $P(\sup_n X_n < \infty) = 0$ which is certainly < 1.

Solution $\Leftarrow$: Now suppose there is a constant M such that

$$\sum_{n=1}^{\infty} P(X_n > M) < \infty$$

By easy half of Borel Cantelli, $P(X_n > M\, i.o.) = 0$. The complement of the event $X_n > M\, i.o$ is not $X_n \leq M\, i.o.$. It is the event that $X_n > M$ for only finite many n. So $P(X_n > M\, finitely\, often) = 1$. Now suppose $X_n > M$ for only finite many n. Let C be the maximum of the X_n which are $> M$. There are only fintely many, so C is finite. (Note that C is a random variable, but that is ok.) We now have $\sup_n X_n \leq \max\{M, C\} = C$ and so $\sup_n X_n < \infty$. So $P(\sup_n X_n < \infty) = 1$.