Math 563 - Fall 15 - Homework 4

1. Let $X_1, X_2, \cdots, X_n$ be RV's and suppose that the distribution of $(X_1, X_2, \cdots, X_n)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$ and the density can be written in the form $g_1(x_1)g_2(x_2)\cdots g_n(x_n)$ where the g_i are non-negative measurable functions but the integral of g_i over $\mathbb{R}$ need not be 1.

(a) Prove that each X_i is an absolutely continuous random variable.

(b) Prove $X_1, X_2, \cdots, X_n$ are independent.

2. Let X be a discrete real-valued random variable, i.e., it has countable range. Let $x_1, x_2, \cdots$ be its values and let $p_n = P(X = x_n)$. Let g be any real valued function on the real line. Suppose that

$$\sum_n |g(x_n)| \, p_n < \infty$$

Prove that $g(X)$ is a random variable and

$$E[g(X)] = \sum_n g(x_n) p_n$$

Note that I did not say that g was measurable.

3. (Resnick) Let X_n be a sequence of **independent** random variables. Consider the random power series $\sum_{n=1}^{\infty} X_n z^n$. Its radius of convergence R is a random variable. Prove that R is a constant almost surely. In other words, prove there is a constant c such that $P(R = c) = 1$. Hint: the radius of convergence R is given by $R^{-1} = \limsup_n |X_n|^{1/n}$. Note that R can be ∞ or 0.

4. (a) Let X and Y be independent random variables which take values in the integers. Prove that the distribution of $X + Y$ is given by

$$P(X + Y = n) = \sum_{m=-\infty}^{\infty} P(X = m) P(Y = n - m)$$

(b) X has a Poisson distribution with parameter $\lambda > 0$ if it takes on the values $0, 1, 2, \cdots$ and

$$P(X = n) = \frac{e^{-\lambda} \lambda^n}{n!}$$

Show that if X and Y are independent random variables, X has a Poisson distribution with parameter λ and Y has a Poisson distribution with parameter μ, then $X+Y$ has a Poisson distribution. What is the parameter for $X+Y$?

5. (Resnick) Let X be a random variable. A number m is called a median of X if $P(X \geq m) \geq 1/2$ and $P(X \leq m) \geq 1/2$. You should convince yourself that there is always at least one median, but it need not be unique. Recall that the mean of X is $\mu = E[X]$ if X is integrable.

(a) Prove that if X has finite variance, then the minimum of $f(a) = E[|X-a|^2]$ occurs at $a = \mu$, the mean.

(b) Let m be a median of X. Prove that the minimum of $g(a) = E[|X-a|]$ occurs at m.