Math 563 - Fall 15 - Homework 5

1. (from Durrett) All the weak and strong laws of large numbers we have seen require that the random variables are independent. In this problem we study relaxing this assumption. Let X_n be identically distributed but not necessarily independent. Assume $EX_n^2 < \infty$. Let $\mu = EX_n$. The covariance of random variables X and Y is $Cov(X,Y) = E[XY] - E[X]E[Y]$. If X and Y are independent then $Cov(X,Y) = 0$, but the converse is not true. Suppose $|Cov(X_n, X_m)| \leq r(|n-m|)$ where $r(k)$ is a function that converges to zero as $k \to \infty$. Let $S_n = X_1 + \cdots X_n$. Prove that $S_n/n \to \mu$ in probability.

2. (from Resnick) Define a sequence of random variables recursively as follows. Let X_0 be uniformly distributed on $[0,1]$. Having defined $X_0, X_1, \cdots, X_n$ we let X_{n+1} be uniformly distributed on $[0, X_n]$. Prove that $\frac{1}{n}\log X_n$ converges a.s. and find the limit.

3. (from Resnick) Let A_n be independent events. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} 1_{A_i} - \frac{1}{n}\sum_{i=1}^{n} P(A_i) \to 0 \, in \, probability$$

4. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. We also assume that the family $\{X_n, Y_n : n = 1, 2, 3, \cdots\}$ is independent. Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0,t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1] + E[Y_1]} \quad a.s.$$

5. We look at the set of all random variables and consider two random variables to be the same if they are equal a.s. Define

$$d(X,Y) = E\left[\frac{|X-Y|}{|X-Y|+1}\right]$$

We proved in class that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$. Prove that with this metric the set of random variables is a complete metric space.