Math 563 - Fall '14 - Homework 6

1. (from Durrett) Let X_n be a sequence of integer valued random variables, X another integer valued random variable. Prove that X_n converge to X in distribution if and only if

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

for all integers m.

2. Take a coin with the probability of heads equal to $p \in (0,1)$ and flip it n times. Let X be the number of heads you get. The distribution of X is called the binomial distribution with n trials. It is easy to show that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots n$$

Now fix a $\lambda > 0$, and let X_n have a binomial distribution with n trials and $p = \lambda/n$. Prove that as $n \to \infty$, $X_n \Rightarrow X$ where X has the Poisson distribution with parameter λ, i.e.,

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

3. (a) Let μ_n be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where $f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that μ_n converges in distribution to μ where μ is $f(x)$ times Lebesgue measure.
(b) Suppose F_n has density $1-\cos(2n\pi x)$ on $[0,1]$ and the density is zero outside this interval. Show that F_n converges weakly to the uniform distribution on $[0,1]$, but that the densities f_n do not converge.

4. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to the random variable c. Prove or disprove each of the following
(a) X_n converges to c in probability
(b) X_n converges to c a.s.

5. (from Durrett, converging together lemma) Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where c is a constant. Prove that $X_n + Y_n \Rightarrow X + c$. Note that this implies that if $X_n \Rightarrow X$ and $Y_n - X_n \Rightarrow 0$, then $Y_n \Rightarrow X$.

6. Let X_n, Y_n, X, Y be random variables defined on the same probability space. Suppose that X_n converges in distribution to X and Y_n converges in distribution to Y. Suppose further that for each n, X_n and Y_n are independent, and that X and Y are independent. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.