Math 563 - Fall '15 - Homework 6 Solutions

1. (from Durrett) Let X_n be a sequence of integer valued random variables, X another integer valued random variable. Prove that X_n converge to X in distribution if and only if

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

for all integers m.

Solution: We prove that if $\lim_{n\to\infty} P(X_n = m) = P(X = m)$ then we have convergence in distribution. The other direction is easy.

For finite a and b, $P(a \le X_n \le b)$ is a finite sum of $P(X_n = m)$. So the hypothesis immediately implies

$$\lim_n P(a \le X_n \le b) = P(a \le X \le b)$$

Now let $\epsilon > 0$. Pick L such that

$$P(-L \le X \le L) \ge 1 - \epsilon$$

Note for later use that this implies $P(X < -L) < \epsilon$. Then pick N so that for $n \ge N$ we have

$$P(-L \le X_n \le L) \ge 1 - 2\epsilon$$

So $P(X_n < -L) < 2\epsilon$ for $n \ge N$. The triangle inequality gives

$$\begin{aligned} & |P(X_n \le b) - P(X \le b)| \\ = \; & |P(X_n < -L) + P(-L \le X_n \le b) - P(-L \le X \le b)| + P(X < -L)| \\ \le \; & |P(-L \le X_n \le b) - P(-L \le X \le b)| + 3\epsilon \end{aligned}$$

Thus

$$\limsup_n |P(X_n \le b) - P(X \le b)| \le 3\epsilon$$

This is true for all $\epsilon > 0$. So the lim sup above is 0, i.e., $P(X_n \le b)$ converges to $P(X \le b)$. So X_n converges to X in disbtribution.

3. (a) Let μ_n be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where

$f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that μ_n converges in distribution to μ where μ is $f(x)$ times Lebesgue measure.

Solution: We first show that f_n converges to f in L^1. Since $f_n \geq 0$, $(f - f_n)^+ \leq f$. (Note that it is not true that $(f - f_n)^- \leq f$.) Since f is integrable and $(f - f_n)^+$ converges to zero a.e., the dominated convergence theorem implies

$$\lim_n \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx = 0$$

Since f_n and f have integral 1,

$$\int_{-\infty}^{\infty} (f(x) - f_n(x)) \, dx = 0$$

So

$$\int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx = \int_{-\infty}^{\infty} (f(x) - f_n(x))^- \, dx$$

Since the left side converges to zero the right side does too. Finally, we have

$$\int_{-\infty}^{\infty} |f(x) - f_n(x)| \, dx = \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx + \int_{-\infty}^{\infty} (f(x) - f_n(x))^- \, dx \to 0$$

Fix an x. Letting F_n, F be the distribution functions of X_n, X,

$$\begin{aligned} |F_n(x) - F(x)| &= |\int_{-\infty}^{x} (f_n(u) - f(u)) \, du| \\ &\leq \int_{-\infty}^{x} |f_n(u) - f(u)| \, du \leq \int_{-\infty}^{\infty} |f_n(u) - f(u)| \, du \end{aligned}$$

The last integral converges to zero, so $F_n(x)$ converges to $F(x)$ for all x.

4. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to the random variable c. Prove or disprove each of the following

(a) X_n converges to c in probability

Solution: This is true. Let F_n be the distribution function of X_n and F the distribution function of c. So $F(x)$ is 0 for $x < c$ and is 1 for $x \geq c$. So the only point where F is not continuous is c. So $F_n(x)$ converges to $F(x)$ for

$x \neq c$. So for any $\epsilon > 0$, $F_n(c+\epsilon) - F_n(c-\epsilon)$ converges to $F(c+\epsilon) - F(c-\epsilon) = 1$. Now

$$F_n(c+\epsilon) - F_n(c-\epsilon) = P(c-\epsilon < X_n \leq c+\epsilon) \geq P(c-\epsilon < X_n < c+\epsilon)$$

So $P(|X_n - c| < \epsilon)$ converges to 1. This shows X_n converges to c in distribution.

(b) X_n converges to c a.s. **Solution:** This is false. By part (a), if X_n converges to c in distribution then it converges in probability. But there are sequences that converge to 0 in probability but do not converge a.s., e. g. the "typewriter sequence."

5. (from Durrett, converging together lemma) Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where c is a constant. Prove that $X_n + Y_n \Rightarrow X + c$. Note that this implies that if $X_n \Rightarrow X$ and $Y_n - X_n \Rightarrow 0$, then $Y_n \Rightarrow X$.

Solution: First note that by a previous problem, since Y_n converges in distribution to a constant c, it converges in probability to this constant.

Let F_n be the distribution function of X_n and F the distribution function of X. Fix an x and let $\epsilon > 0$. We find upper and lower bounds on $P(X_n + Y_n \leq x)$. We get an upper bound as follows.

$$\begin{aligned}
& P(X_n + Y_n \leq x) \\
= \; & P(X_n + Y_n \leq x, |Y_n - c| \leq \epsilon) + P(X_n + Y_n \leq x, |Y_n - c| > \epsilon) \\
\leq \; & P(X_n \leq x - c + \epsilon) + P(|Y_n - c| > \epsilon)
\end{aligned}$$

We know that $P(|Y_n - c| > \epsilon)$ converges to 0 as $n \to \infty$. If $x - c + \epsilon$ is a continuity point of F, then $P(X_n \leq x - c + \epsilon) = F_n(x - c + \epsilon)$ converges to $F(x - c + \epsilon)$. So

$$\limsup_n P(X_n + Y_n \leq x) \leq F(x - c + \epsilon)$$

for all ϵ such that $x - c + \epsilon$ is a continuity point of F. We can find a sequence of such ϵ which converge to 0, and $F(x - c + \epsilon)$ converges to $F(x - c)$ as $\epsilon \to 0$. So

$$\limsup_n P(X_n + Y_n \leq x) \leq F(x - c)$$

We get a lower bound as follows.

$$P(X_n \leq x - c - \epsilon)$$

$$
\begin{aligned}
&= P(X_n \leq x - c - \epsilon, |Y_n - c| < \epsilon) + P(X_n \leq x - c - \epsilon, |Y_n - c| \geq \epsilon) \\
&\leq P(X_n + Y_n \leq x) + P(|Y_n - c| \geq \epsilon)
\end{aligned}
$$

So if $x - c - \epsilon$ is a continuity point of F, then taking the $\liminf$ we get

$$\liminf_n P(X_n + Y_n \leq x) \geq F(x - c - \epsilon)$$

If $x - c$ is a continuity point of F, then letting $\epsilon \to 0$ we get

$$\liminf_n P(X_n + Y_n \leq x) \geq F(x - c)$$

Our two bounds imply

$$\lim_n P(X_n + Y_n \leq x) = F(x - c)$$

Since $P(X + c \leq x) = P(X \leq x - c) = F(x - c)$, this shows $X_n + Y_n$ converges to $X + c$ in distribution.