**Math 563 - Fall '15 - Homework 7**

1. The Cauchy distribution has the density function

$$f(x) = \frac{1}{\pi(1+x^2)}$$

I.e., the distribution is Lebesgue measure on the real line times this function.
(a) Compute the characteristic function of a Cauchy random variable. Hint: contour integration.
(b) Show that if $X_n$ is an i.i.d. sequence with the Cauchy distribution, then for all $n$, $\frac{1}{n}\sum_{k=1}^{n} X_k$ has the Cauchy distribution.

2. Suppose that for each $n$, $X_n$ has a normal distribution with mean 0 and variance $\sigma_n^2$. Suppose there is a random variable $X$ such that $X_n \Rightarrow X$. Prove $\sigma_n^2$ converges to some constant $\sigma \in [0, \infty)$ and that $X$ is normal with variance $\sigma^2$. (Normal with variance 0 should be interpretted to mean that $X = 0$.)

3. **Renewal theory** (from Resnick) Suppose $X_n$ are i.d.d. non-negative random variables. Let $\mu$ be their mean and $\sigma^2$ their variance. Let

$$N(t) = \sup\{n : S_n \leq t\}$$

Prove that

$$\frac{N(t) - \frac{t}{\mu}}{\sigma t^{1/2} \mu^{-3/2}} \Rightarrow Z$$

where $Z$ has a standard normal distribution.

4. (from Durrett) Let $X_n$ be independent, and suppose there is a constant $M$ such that $|X_n| \leq M \, a.s.$ for all $n$. Suppose also that $\sum_n var(X_n) = \infty$. Let $S_n = X_1 + \cdots + X_n$. Prove that

$$\frac{S_n - ES_n}{\sqrt{var(S_n)}} \Rightarrow Z$$

where $Z$ has a standard normal distribution.

5. The gamma distribution is a two parameter family of densities. The parameters $\lambda$ and $w$ are both positive. The density is

$$f(x) = \frac{\lambda^w}{\Gamma(w)} x^{w-1} \exp(-\lambda x)$$

for $x \geq 0$. The density is 0 for $x < 0$. The gamma function is defined by

$$\Gamma(w) = \int_0^\infty x^{w-1} \, e^{-x} \, dx$$

Integration by parts shows $\Gamma(w+1) = w\Gamma(w)$. If $w$ is an integer then $\Gamma(w) = (w-1)!$. A little calculus shows the mean of the gamma distribution is $w/\lambda$ and the variance is $w/\lambda^2$.

(a) Find the characteristic function of the gamma distribution.

(b) For positive integers $n$ let $X_n$ be a random variable with a gamma distribution with $w = n$ and $\lambda = 1$. Prove that $(X_n - n)/\sqrt{n}$ converges in distribution to a standard normal.