Math 563 - Fall 18 - Homework 1
Due Wed, Sept 5

1. Let E_n be a sequence of events. We define a new event :

$$\{\omega : \exists\ infinite\ I \subset \mathbb{N}\ such\, that\ i \in I \Rightarrow \omega \in E_i\}$$

This event is sometimes written $E_n\, i.o.$, where $i.o.$ stands for "infinitely often."
(a) Show that $E_n\, i.o. = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$
(b) Prove that if $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P(E_n\, i.o.) = 0$. This is sometimes called the "easy half" of the Borel Cantelli lemma.

2. (a) Let X_1 have an exponential distribution with parameter $\lambda = 1$, i.e., $F_{X_1}(x) = 1 - e^{-x}$. Let X_2 have an exponential distribution with parameter $\lambda = 2$, i.e., $F_{X_2}(x) = 1 - e^{-2x}$. We define a new RV X as follows. We flip a coin. If it is heads, we set $X = X_1$, it it is tails we set $X = X_2$. Find the distribution of X.

(b) Let Y be a random variable which is uniform on $[0, 1]$, i.e., the distribution μ_Y is Lebesgue measure on $[0, 1]$. Define a RV X by

$$X = \begin{cases} Y & \text{if } Y \leq 1/2 \\ 1, & \text{if } Y > 1/2 \end{cases}$$

NOTE the correction in the above since the homework was first posted. Find the distribution of X.

3. Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$. This is problem 1.3.8 in Durrett. For a hint look at problem 1.3.7 or 1.3.9.

4. (loosely based on a problem in Resnick) Let P_1, P_2 be two probability measures on $(\Omega, \mathcal{F})$. You might think that if they agree on a collection $\mathcal{C}$ of events, and the σ-field generated by $\mathcal{C}$ is $\mathcal{F}$, then the two probability measures agree on all of $\mathcal{F}$. This is not true. The point of this problem is to give a counterexample. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$

(a) What is the σ-field $\mathcal{F}$ generated by $\mathcal{C}$?

(b) Find two probability measures P_1 and P_2 which agree on $\mathcal{C}$ but do not agree on $\mathcal{F}$.

5. (from Resnick) Let P be a probability measure on $\mathcal{B}(\mathbb{R})$, the Borel sets in $\mathbb{R}$. Prove that for any $E \in \mathcal{B}(\mathbb{R})$ and any $\epsilon > 0$ there exits a finite union of disjoint intervals A such that $P(E \Delta A) < \epsilon$.

Hint: Define $\mathcal{F}$ to be the collection of Borel sets such that for all $\epsilon > 0$ there exits a finite union of disjoint intervals A such that $P(E \Delta A) < \epsilon$. What can you prove about $\mathcal{F}$?

Do one of the problems labelled 6. below. The second one is much more interesting, but harder.

6. We flip a fair coin infinitely many times. Let X_n be 1 if the nth flip is heads, and 0 if the nth flip is tails. The sample space Ω consists of all sequences of heads and tails. X_n is a real valued function on Ω. In this problem we assume that there is a σ-field $\mathcal{F}$ and a probability measure P such that X_n is a random variable and the probability measure agrees with your intiution. (We will eventually prove such an $\mathcal{F}$ and P exist.) Define

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

Note that $0 \le X \le 1$. Find the distribution μ_X of X. Hint: find $P(X \in E)$ when E is an interval of the form $((k-1)/2^n, k/2^n)$ for integers k and n.

6. Let X_n be as in the last problem. Now define

$$Y = \sum_{n=1}^{\infty} \frac{2X_n}{3^n}$$

NB: It is 3^n in the denominator, not 2^n.

(a) Prove that the distribution function F_Y is continuous.

(b) Prove that F_Y is differentiable a.e. with the derivative equal to 0 a.e. Hint: prove that F_Y is constant on the complement of the Cantor set.

(c) Let μ_Y be the distribution of Y. Let m be Lebesgue measure on the real line. Prove that μ_Y and m are mutually singular. This means that there is a Borel set A with $m(A) = 0$ and $\mu_Y(A^c) = 0$.