Math 563 - Fall 18 - Homework 1 - Solution

3. Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

Solution: We prove that if Y is $\sigma(X)$ measurable, then there is a Borel measurable f such that $Y = f(X)$. The other direction is easy.

For each postive integer n, we define

$$E_{n,k} = \{\omega : \frac{k-1}{2^n} < Y(\omega) \leq \frac{k}{2^n}\}$$

where k ranges over all integers. Note that the $E_{n,k}$ are disjoint and their union is all of Ω. Since Y is $\sigma(X)$ measurable, there are Borel sets $B_{n,k}$ in $\mathbb{R}$ such that $E_{n,k} = X^{-1}(B_{n,k})$. Note that the $B_{n,k}$ need not be disjoint since they can contain points that are not in the range of X. It is tempting to make them disjoint by replacing $B_{n,k}$ by its intersection with the range of X. But the range of X need not be a Borel set, so you cannot do this.

Define functions on $\mathbb{R}$ by

$$f_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} 1_{B_{n,k}}$$

We claim that for all n and all ω, we have

$$|f_n(X(\omega)) - Y(\omega)| < 2^{-n}$$

Let k be such that $\omega \in E_{n,k}$. So $(k-1)2^{-n} < Y(\omega) \leq k2^{-n}$. Since $\omega \in X^{-1}(B_{n,k})$, $X(\omega) \in B_{n,k}$. If $l \neq k$, $\omega \notin E_{n,l}$. So $\omega \notin X^{-1}(B_{n,l})$. So $X(\omega) \notin B_{n,l}$ for $l \neq k$. So $f_n(X(\omega)) = k2^{-n}$. This proves the claim.

The claim implies that $f_n(x)$ converges if x is in the range of X. And if $x = X(\omega)$ then $f_n(x)$ converges to $Y(\omega)$. However, outside the range of X we don't know anything about f_n. Let L be the set of real numbers where f_n converges. Since the f_n are Borel measurable, L is a Borel set. And L contains the range of X. So we can define f to be the limit of f_n on L and define f to be zero outside L. Then f is Borel measurable. And the claim above implies $f(X) = Y$.