Math 563 - Fall 18 - Homework 2

1. (based on a problem in Resnick) Let X_n be a sequence of RV's on $(\Omega, \mathcal{F}, P)$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

Let $\tau = \inf\{n > 0 : S_n > 0\}$. The inf of an empty set is defined to be ∞, so τ can take on the value ∞.
(a) Prove τ is a random variable, i.e., it is measurable. Note that τ takes values in the extended reals, so by measurable I mean measurable with respect to the Borel sets in the extended reals.
(b) Define X to be S_τ when τ is finite and 0 when $\tau = \infty$. Prove X is a random variable.

2. (from Durrett) Suppose $EY = 0$ and the variance σ^2 of Y is finite. Let $a > 0$. Prove

$$P(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Hint: apply Chebyshev with $\phi(y) = (y + b)^2$ and optimize your result over b.

3. Prove that a random variable X is independent of itself if and only if there is a constant c such that $P(X = c) = 1$. Hint: what can you say about the distribution function of X?

4. (from Durrett)
(a) Let X and Y be independent random variables which take values in the integers. Prove that the distribution of $X + Y$ is given by

$$P(X + Y = n) = \sum_{m=-\infty}^{\infty} P(X = m) P(Y = n - m)$$

(b) X has a Poisson distribution with parameter $\lambda > 0$ if it takes on the values $0, 1, 2, \cdots$ and

$$P(X = n) = \frac{e^{-\lambda} \lambda^n}{n!}$$

Show that if X and Y are independent random variables, X has a Poisson distribution with parameter λ and Y has a Poisson distribution with parameter μ, then $X + Y$ has a Poisson distribution. What is the parameter for $X + Y$?

5. (from Durrett) Let $\Omega = (0,1)$, $\mathcal{F}$ be the Borel sets in $(0,1)$, and let P be Lebesgue measure. Define RV's by

$$X_n(\omega) = \begin{cases} 0 & \text{if } [2^n \omega] \text{ is even} \\ 1 & \text{if } [2^n \omega] \text{ is odd} \end{cases}$$

where $[x]$ is the largest integer less than or equal to x. Sketch a proof that the $\{X_n\}_{n=1}^{\infty}$ are independent random variables. I say "sketch" because writing out all the details for this can get out of control. Note that this gives a rigorous construction of the probability space for flipping a fair coin infinitely many times.

The change of variables theorem says if X is a random variable and f a real-valued measurable function on the real line (with some condition on f), then

$$E[f(X)] = \int_{\mathbb{R}} f(x)\, d\mu_X$$

where μ_X is the distribution of X. The point of the next two problems is to get more explicit formulae that are amenable to computation in the cases that X is discrete or absolutely continuous.

6. Let X be a discrete real-valued random variable. Recall that this means there is a finite or countable set of values $\{x_n\}_{n=1}^{N}$ with $P(X = x_n \, for\, some\, n) = 1$. (Here N can be finite or ∞.) Let $p_n = P(X = x_n)$. Let g be any real valued function on the real line. Suppose that

$$\sum_{n=1}^{N} |g(x_n)|\, p_n < \infty$$

Prove that $g(X)$ is a random variable and

$$E[g(X)] = \sum_{n=1}^{N} g(x_n) p_n$$

Note that I did not say that g was measurable.

7. Let X be a random variable whose distribution is absolutely continuous with respect to Lebesgue measure. So there is a density function $f(x)$ so

that the distribution of X is $f(x)$ times Lebesgue measure. Let $g(x)$ be a real-valued measurable function on the real line such that

$$\int_{-\infty}^{\infty} |g(x)| \, f(x) \, dx < \infty$$

Prove that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx$$

where the integrals with respect to dx are integration with respect to Lebesgue measure.

The last three problems are not to be turned in. They are standard problems in a course like 564. I have included them for background.

8. (564 discrete RV's) Let X be a discrete RV, $x_1, x_2, \cdots$ its values and $p_n = P(X = x_n)$. The function $f(x)$ which is p_n at x_n and is 0 at points not equal to one of the x_n is often called the "probability mass function" in 564 level probability courses.

Suppose we have a coin with probability p of heads. (p is not necessarily 1/2.) The following discrete RV's are of interest:

Binomial: Fix a positive integer n. Flip the coin n times and let X be the number of heads. So X can be $0, 1, 2, \cdots, n$.

Geometric: Flip the coin until you get heads for the first time. Let X be the number of tails. (Warning: depending on the author the definition of X is sometimes taken to be the total number of flips, including the final one that gave heads.) With our definition of X the possible values of X are the nonnegative integers.

Negative binomial: Fix an integer r. Flip the coin until we get heads for the rth time. Let X be the total number of tails. So the possible values of X are the nonnegative integers.

For each of these find the probability mass function and the expected value of X.

9. (564 continuous RV's) In a 564 level course, "continuous RV" usually means that the distribution of the RV is absolutely continuous with respect to Lebesgue measure and so is given by $f(x)dx$. (Having a density implies the distribution function is continuous, but the converse is very false.) Here are three common "continuous" RV's.

Uniform: We say X is uniform on $[a, b]$ if $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and $f(x) = 0$ otherwise.
Exponential: We say X has an exponential distribution if $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Here λ is a positive parameter.
Normal: We say X has a normal distribution if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Here μ is a real parameter and σ^2 is a positive parameter.

Find the expected value of each of these random variables.

10. Let X be a random variable with a normal distribution. Find the density function of X^2. Hint: first find an expression for the distribution function of X^2.