Math 563 - Fall 18 - Homework 3

1. (from Durrett) Let $X_1, X_2, \cdots, X_n$ be RV's and suppose that the distribution of $(X_1, X_2, \cdots, X_n)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$ and the Radon-Nikodym derivative is $g_1(x_1)g_2(x_2)\cdots g_n(x_n)$ where the g_i are non-negative measurable functions but need not have integral 1. Prove $X_1, X_2, \cdots, X_n$ are independent.

2. (from Durrett) *metric for convergence in probability:* We look at the set of all random variables and consider two random variables to be the same if there are equal a.s. Define

$$d(X,Y) = E\left[\frac{|X-Y|}{|X-Y|+1}\right]$$

(a) It is obvious that d is reflexive and $d(X,Y) = 0$ if and only if $X = Y$ a.s. Prove that d satisfies the triangle inequality and so defines a metric.
(b) Prove that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$.
(c) **Optional** Prove that this metric space is complete.

3. Let $X_1, \cdots, X_n$ be random variables and let $Y_1, \cdots, Y_n$ be random variables. Suppose that their joint distribution functions are equal:

$$F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = F_{Y_1,\cdots,Y_n}(x_1, \cdots, x_n)$$

Use the $\pi - \lambda$ theorem to prove that their joint distributions are equal, i.e.,

$$\mu_{X_1,\cdots,X_n} = \mu_{Y_1,\cdots,Y_n}$$

where the two measures are Borel measures on $\mathbb{R}^n$. There is a theorem that says a Borel measure on $\mathbb{R}^n$ is determined by its values on the rectangles. You should not use this theorem.

4. (from Durrett) Let X and Y be independent RV's with distributions μ and ν.
(a) Prove that

$$P(X+Y=0) = \sum_x \mu(\{x\})\,\nu(\{-x\})$$

(b) Use the previous part to prove that if we also know that $P(X = x) = 0$ for all x, then $P(X = Y) = 0$.

5. (from Durrett) Let $\Omega = \{1, 2, 3, 4\}$. Let $\mathcal{F}$ be all subsets of Ω. Define P by $P(\{i\}) = 1/4$ for $i = 1, 2, 3, 4$. Give an example of two collections $\mathcal{A}_1$, $\mathcal{A}_2$ of events which are independent but such that $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are not independent.

6. Suppose X_n are identically distributed, uncorrelated, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

(b) Now suppose that you also know that $EX_n^4 < \infty$ and the X_n are independent. Prove that $P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p.