Math 563 - Fall 18 - Homework 4

1. (from Durrett) *Monte Carlo integration:* Let $f(x)$ be a Lebesgue integrable function on $[0,1]$. Let U_n be an i.i.d. sequence with each U_n uniformly distributed on $[0,1]$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} f(U_k) \to \int_0^1 f(x)\,dx$$

with probability one. This is a really bad way to numerically compute a one-dimensional integral, but there are analogs of this in higher dimension that are sometimes one of the best ways to compute high dimensional integrals.

2. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. (We also assume $\{X_n, Y_n : n = 1, 2, 3, \cdots\}$ is independent.) Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0,t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1] + E[Y_1]} \quad a.s.$$

3. (from Durrett) Let X_n be an independent sequence of RV's. Prove that $sup_n X_n < \infty$ a.s. if and only if there is a constant C such that

$$\sum_{n=1}^{\infty} P(X_n > C) < \infty.$$

4. (from Durrett) Let X_n be independent RV's. X_n has a Poisson distribution with $EX_n = \lambda_n$. Suppose that $\sum_n \lambda_n = \infty$. Define $S_n = X_1 + \cdots X_n$. Prove that

$$\frac{S_n}{ES_n} \to 1 \quad a.s.$$

Hint: prove it for a subsequence and then "sandwich".

5. (from Durrett) Let A_n be a sequence of independent events with $P(A_n) < 1$ for all n. Prove

$$P(\cup_{n=1}^{\infty} A_n) = 1 \, if \, and \, only \, if \, P(A_n \, i.o.) = 1$$