Math 563 - Fall '18 - Homework 5

Do 5 of the 6 problems

1. (from Durrett, converging together lemma) Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where c is a constant. Prove that $X_n + Y_n \Rightarrow X + c$. Note that this implies that if $X_n \Rightarrow X$ and $Y_n - X_n \Rightarrow 0$, then $Y_n \Rightarrow X$.

2. The gamma distribution is a two parameter family of densities. The parameters λ and w are both positive. The density is

$$f(x) = \frac{\lambda^w}{\Gamma(w)} x^{w-1} \exp(-\lambda x)$$

for $x \geq 0$. The density is 0 for $x < 0$. The gamma function is defined by

$$\Gamma(w) = \int_0^\infty x^{w-1} e^{-x} \, dx$$

Integration by parts shows $\Gamma(w+1) = w\Gamma(w)$. If w is an integer then $\Gamma(w) = (w-1)!$. A little calculus shows the mean of the gamma distribution is w/λ and the variance is w/λ^2.

For positive integers n let X_n be a random variable with a gamma distribution with $w = n$ and $\lambda = 1$. Prove that $(X_n - n)/\sqrt{n}$ converges in distribution to a standard normal.

3. Suppose that X_n converges in distribution to X and Y_n converges in distribution to Y. Suppose further that for each n, X_n and Y_n are independent, and that X and Y are independent. Assume that X_n, Y_n, X, Y are defined on the same probability space. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

4. Let X_n be i.i.d. with $EX_n = 0$ and $EX_n^2 = 1$. Let $S_n = \sum_{k=1}^n X_k$. Use the central limit theorem and the Kolmogorov 0-1 law to prove that

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = \infty \quad a.s.$$

5. Let X_n be an i.i.d. sequence with $EX_n = \mu$ and $Var(X_n) = \sigma^2 < \infty$. The sample mean is defined by

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

By the strong law of large numbers it converges a.s. to μ. The sample variance is defined by

$$V_n = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2$$

One of the midterm problems shows it converges a.s. to σ^2.

Prove that

$$\frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{nV_n}}$$

converges in distribution to the standard normal distribution.

6. The Cauchy distribution has the density function

$$f(x) = \frac{1}{\pi(1+x^2)}$$

(a) Compute the characteristic function of a Cauchy random variable. Hint: contour integration.

(b) Show that if X_n is an i.i.d. sequence with the Cauchy distribution, then for all n, $\frac{1}{n} \sum_{k=1}^{n} X_k$ has the Cauchy distribution. Note that the CLT does not hold here. (The variance is infinite. In fact, the distribution does not even have a first moment.)