Math 563 - Fall '18 - Homework 6

1. (Thinning a Poisson distribution) Let N have a Poisson distribution with parameter λ. (Note N is not a Poission process, just a single RV.) Let X_n be an i.i.d. sequence taking only the values 0 and 1. Suppose the RV's N and all the X_n are independent. Let

$$N_0 = |\{n \leq N : X_n = 0\}|, \qquad N_1 = |\{n \leq N : X_n = 1\}|$$

Here $| \quad |$ just means the cardinality of the set. Prove that N_0 and N_1 are independent, and each is Poisson with parameters $(1-p)\lambda$ and $p\lambda$ where $p = P(X_n = 1)$.

2. (Thinning a Poisson process) Let N_t be a Poisson process. Let X_n be an i.i.d. sequence which only takes on the values 0 and 1. Assume that all the RV's N_t and X_n are independent. Define a new process M_t by

$$M_t = |\{n \leq N_t : X_n = 0\}|$$

Prove that M_t is a Poisson process.

3. (based on Durrent 3.10.4) Let X_n and X be random vectors (values in $\mathbb{R}^d$). Suppose that X_n converges in distribution to X. Let $X_n = (X_n^1, \cdots, X_n^d)$ and $X = (X^1, \cdots, X^d)$.
(a) Prove that for each i the components X_n^i converge in distribution to X^i.
(b) Prove that the converse to (a) is false by giving a counterexample.

4. (based on Durrent 3.10.6 and 3.10.7)

(a) Let $X = (X_1, \cdots, X_d)$ be be a random vector. Let $\phi_X(t)$ be the characteristic function of X. (So it is a function on $\mathbb{R}^d$.) Let $\phi_{X_i}(t)$ be the characteristic function of X_i. (So it is a function on $\mathbb{R}$.) Prove that $X_1, X_2, \cdots, X_d$ are independent if and only if

$$\phi_X(t_1, t_2, \cdots, t_d) = \prod_{i=1}^{d} \phi_{X_i}(t_i)$$

(b) Let $X = (X_1, \cdots, X_d)$ have a multivariate normal distribution with covariance matrix C_{ij}. Prove that $X_1, X_2, \cdots, X_d$ are independent if and only the matrix C is diagonal.

5. (Durrett 3.4.10) Let X_n be independent but not necessarily identically distributed. Assume there is a constant M such that $|X_n| \leq M \, a.s.$ for all n. Assume also that $\sum_{n=1}^{\infty} var(X_n) = \infty$. Let $S_n = \sum_{k=1}^{n} X_k$. Prove that

$$\frac{S_n - ES_n}{\sqrt{var(S_n)}} \Rightarrow Z$$

where Z has a standard normal distribution.