Math 563 - Fall '18 - Homework 7

Do 4 of the 5 problems

1. Let X_n be an i.i.d. sequence with a symmetric distibution. Suppose the distribution measure is absolutely continuous with respect to Lebesgue measure and the density is continuous and not equal to zero at 0. For $p > 0$, consider

$$\frac{1}{n^\beta} \sum_{j=1}^{n} \frac{1}{X_j |X_j|^{p-1}}$$

Show that it converges in distribution for a suitable choice of β (depending on p) and determine the limiting distribution. (It will of course depend on p.) Hint: we did the case of $p = 1$ as an example in class. Note that if X has a symmetric distibution, then so does $\frac{1}{X|X|^{p-1}}$.

2. (from Durrett) Let X_n be i.i.d. with a Poisson distribution with mean 1. Large deviation theory says

$$\lim_n \frac{1}{n} ln(P(S_n \geq an)) = \gamma(a)$$

for $a > 1$. Find the function $\gamma(a)$. It is also possible to do this without large deviation theory. See Durrett exercise 3.1.4, where you will also find the answer.

3. Let X and Y be independent random variables. X has a Poisson distribution with mean $E[X] = \lambda$ and Y has a Poisson distribution with mean $E[Y] = \mu$. Let $Z = X + Y$. Find $E[X|Z]$ and $E[Z|X]$. (One of these is easy.)

4. (from Durrett) Let $\Omega = \{a, b, c\}$. Given an example of a random variable X and σ-fields $\mathcal{F}_1$,$\mathcal{F}_2$ such that

$$E[E[X|\mathcal{F}_1]|\mathcal{F}_2] \neq E[E[X|\mathcal{F}_2]|\mathcal{F}_1]$$

5. (from Durrett) Define $var[X|\mathcal{F}]$ to be $E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$. Prove that

$$var(X) = E[var[X|\mathcal{F}]] + var(E[X|\mathcal{F}])$$

(This is short).