Math 563 - Fall '21 - Final (take home)

Due : You can work on the final for one week (7 consecutive days). It is due at the end of that week, but no later than Wed, Dec 15 at 11:59pm. Upload your solutions to gradescope.

The fine print: You are supposed to do this exam on your own. This means you should not talk to anyone about the exam. You can ask me questions, but I will only answer questions about what the problem is asking, not about how to do it. You can consult your class notes, past homeworks and Durrett. You cannot consult any other sources including the web.

Do 5 of the 6 problems. Do not turn in 6 problems.

1. Let X_n be an i.i.d. sequence of random variables. Each X_n has an exponential distribution with parameter λ. So its density is $\lambda \exp(-\lambda x)$ for $x \geq 0$. Prove that

$$\limsup_{n\to\infty} \frac{X_n}{\ln n} = \frac{1}{\lambda} \ a.s.$$

Hint: Given a $c > 0$, what can you say about $P(\frac{X_n}{\ln n} \geq c \ i.o.)$.

2. Let A_n be a sequence of independent events such that $\lim_{n\to\infty} P(A_n)$ exists and is not 0 or 1. Prove that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} [1_{A_k} - P(A_k)]$$

converges in distribution to a normal random variable and find the mean and variance of this limiting normal random variable.

3. Let X_n and X be random variables on the same probability space. Assume that $E[|X_n|] < \infty$ and $E[|X|] < \infty$.

Here are four types of convergence: (1) pointwise convergence almost surely, (2) convergence in probability, (3) convergence in distribution, and (4) convergence in L^1, i.e., $E|X_n - X|$ converges to zero.

(a) For each of the four types of convergence prove or disprove that if X_n converges to X then $E[X_n]$ converges to $E[X]$.

(b) For each of the four types of convergence prove or disprove that if X_n converges to X then $E[\cos(X_n)]$ converges to $E[\cos(X)]$.

4. A random variable X has a geometric distribution if it takes values in the non-negative integers and $P(X = k) = (1-p)^k p, k = 0, 1, 2, \cdots$. Here p is a parameter between 0 and 1. Suppose you have a coin with probability of heads equal to p. Flip it until you get heads for the first time. Let X be the total number of flips. Then X has a geometric distribution.

Let $\alpha > 0$. Let X_n be a sequence of random variables such that X_n has the geometric distribution with $p = \alpha/n$. Let $Y_n = X_n/n$. Prove that Y_n converges in distribution and find the limiting distribution. A list of some facts about common discrete and continuous distributions is on the next page and can be used for this problem. Note that it gives the moment generating function $M(t) = E[\exp(tX)]$ rather than the characteristic function.

5. Let ξ_n be i.i.d. RV's. We assume they are integrable. Let $X_n = \sum_{k=1}^n \xi_k$. Assume that $E[\xi_n] \neq 0$. Recall that this implies X_n is not a martingale. Define $Z_n = \exp(\alpha X_n)$ where α is a constant.

(a) Show that if there is an α such that $\exp(\alpha \xi_n)$ is integrable and

$$E[\exp(\alpha \xi_n)] = 1$$

then Z_n is a martingale.

(b) Now suppose ξ_n only takes on the values 1 and -1 and let $p = P(\xi_n = 1)$. Now X_n is a non-symmetric random walk. Let a, b be integers with $a < 0 < b$. We start the walk at 0 and X_n is the position at time n. Let τ be the time when we first hit a or b. Assume that the hypotheses of the optional stopping theorem are satisfied. Find the probability the walk hits a before it hits b, i.e., find $P(X_\tau = a)$.

6. Let X be a random variable with $E[X^2] < \infty$. The conditional variance of X given $\mathcal{F}$ is the random variable defined by

$$Var(X|\mathcal{F}) = E[X^2|\mathcal{F}] - (E[X|\mathcal{F}])^2$$

Let X_k be an independent sequence of random variables with $E[X_k^2] < \infty$. Define $S_n = \sum_{k=1}^n X_k$, and let $\mathcal{F}_n$ be the σ-field generated by $X_1, X_2, \cdots, X_n$. Prove that $Var(S_{n+1}|\mathcal{F}_n) = Var(X_{n+1})$ a.s.

Some popular RV's

μ is the mean, σ^2 is the variance, $M(t) = E[e^{tX}]$.

Binomial (2 parameters, $p \in [0,1]$ and positive integer n) :

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

$$\mu = np, \quad \sigma^2 = np(1-p), \quad M(t) = (pe^t + 1 - p)^n$$

Geometric (1 parameter $p \in (0,1]$) :

$$P(X = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}, \quad M(t) = \frac{pe^t}{1-(1-p)e^t}$$

Poisson (1 parameter $\lambda > 0$) :

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots \qquad \mu = \lambda, \quad \sigma^2 = \lambda, \quad M(t) = \exp(\lambda(e^t - 1))$$

Exponential (1 parameter, $\lambda > 0$) :

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \qquad \mu = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}, \quad M(t) = \frac{\lambda}{\lambda - t}$$

Normal (2 parameters, μ and $\sigma^2 > 0$) :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad M(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$$

Gamma (2 parameters, $\lambda > 0$, $w > 0$):

$$f(x) = \frac{\lambda^w}{\Gamma(w)} x^{w-1} e^{-\lambda x}, \quad x \geq 0$$

$$\mu = \frac{w}{\lambda}, \quad \sigma^2 = \frac{w}{\lambda^2}, \quad M(t) = \left(\frac{\lambda}{\lambda - t}\right)^w$$