Math 563 - Fall 21 - Homework 1
Due Wed., Sept 8

1. (a) Let X_1 have an exponential distribution with parameter $\lambda = 1$, i.e., $F_{X_1}(x) = 1 - e^{-x}$. Let X_2 have an exponential distribution with parameter $\lambda = 2$, i.e., $F_{X_2}(x) = 1 - e^{-2x}$. We define a new RV X as follows. We flip a coin. If it is heads, we set $X = X_1$, it it is tails we set $X = X_2$. Find the distribution function F_X of X.

(b) Let Y be a random variable which is uniform on $[0, 1]$, i.e., the distribution μ_Y is Lebesgue measure on $[0, 1]$. Define a RV X by

$$X = \begin{cases} Y & \text{if } Y \leq 1/2 \\ 1, & \text{if } Y > 1/2 \end{cases}$$

Find the distribution function F_X of X, and the distribution meaure μ_X.

2. (loosely based on a problem in Resnick) Let P_1, P_2 be two probability measures on $(\Omega, \mathcal{F})$. You might think that if they agree on a collection $\mathcal{C}$ of events, and the σ-field generated by $\mathcal{C}$ is $\mathcal{F}$, then the two probability measures agree on all of $\mathcal{F}$. This is not true. The point of this problem is to give a counterexample. Let $\Omega = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$

(a) What is the σ-field $\mathcal{F}$ generated by $\mathcal{C}$?

(b) Find two probability measures P_1 and P_2 which agree on $\mathcal{C}$ but do not agree on $\mathcal{F}$.

3. Let E_n be a sequence of events. We define a new event :

$$\{\omega : \exists\ infinite\ I \subset \mathbb{N}\ such\, that\ i \in I \Rightarrow \omega \in E_i\}$$

This event can be thought of as the event that "E_n happens infinitely often, and so is sometimes written $E_n\, i.o.$.

(a) Show that $E_n\, i.o. = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$

(b) Prove that if $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P(E_n\, i.o.) = 0$. This is sometimes called the "easy half" of the Borel Cantelli lemma.

4. (from Durrett) Let $\Omega = \mathbb{R}$. Let $\mathcal{F}$ contain all the sets that are finite or countable or whose complement is finite or countable. For $A \in \mathcal{F}$, define $P(A) = 0$ is A is finite or countable and $P(A) = 1$ is A^c is finite or countable. Prove $\mathcal{F}$ is a σ-field and that P is a probability measure on it.

Do one of the problems labelled 5. below. The second one is more interesting, but much harder. It assumes some familiarity with the Cantor set

5. We flip a fair coin infinitely many times. Let X_n be 1 if the nth flip is heads, and 0 if the nth flip is tails. The sample space Ω consists of all sequences of heads and tails. X_n is a real valued function on Ω. In this problem we assume that there is a σ-field $\mathcal{F}$ and a probability measure P such that X_n is a random variable and the probability measure agrees with your intiution. (We will eventually prove such an $\mathcal{F}$ and P exist.) Define

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

Note that $0 \leq X \leq 1$. Find the distribution μ_X of X. Hint: find $P(X \in E)$ when E is an interval of the form $((k-1)/2^n, k/2^n)$ for integers k and n.

5. Let X_n be as in the last problem. Now define

$$Y = \sum_{n=1}^{\infty} \frac{2X_n}{3^n}$$

NB: It is 3^n in the denominator, not 2^n.

(a) Prove that the distribution function F_Y is continuous.

(b) Prove that F_Y is differentiable a.e. with the derivative equal to 0 a.e. Hint: prove that F_Y is constant on the complement of the Cantor set.

(c) Let μ_Y be the distribution of Y. Let m be Lebesgue measure on the real line. Prove that μ_Y and m are mutually singular. This means that there is a Borel set A with $m(A) = 0$ and $\mu_Y(A^c) = 0$.