Math 563 - Fall 21 - Homework 2

1. (from Resnick, slightly modified) Let X_n be a sequence of RV's on $(\Omega, \mathcal{F}, P)$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

Let $\tau = \inf\{n > 0 : S_n > 0\}$. Note that τ may sometimes be ∞.
(a) Prove τ is a random variable, i.e., it is measurable. Note that τ takes values in the extended reals, so by measurable I mean measurable with respect to the Borel sets in the extended reals. Hint: note that τ only takes on a countable number of values (the positive integers and ∞). First prove that it is enough to show that $\tau^{-1}(n) \in \mathcal{F}$ when n is one of these values.
(b) (optional - will not be graded) Define X to be S_τ when τ is finite and 0 when $\tau = \infty$. Prove X is a random variable.

2. (from Durrett) Suppose $EY = 0$ and the variance σ^2 of Y is finite. Let $a > 0$. Prove

$$P(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Hint: apply Chebyshev with $\phi(y) = (y+b)^2$ and optimize your result over b.

3. Let $X_1, X_2, \cdots, X_n$ be RV's such that the joint distribution $\mu_{X_1, X_2, \cdots, X_n}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. So there is a measurable function $f(x_1, x_2, \cdots, x_n)$ such that for Borel sets A in $\mathbb{R}^n$

$$P((X_1, X_2, \cdots, X_n) \in A) = \int_A f(x_1, x_2, \cdots, x_n) dx_1 \cdots dx_n$$

Suppose there are non-negative measurable functions $g_1, g_2, \cdots, g_n$ on $\mathbb{R}$ such that

$$f(x_1, x_2, \cdots, x_n) = g_1(x_1) g_2(x_2) \cdots g_n(x_n)$$

Prove that $X_1, X_2, \cdots, X_n$ are independent. Note that it is not assumed that the g_i have integral equal to 1.

Do one of the two problems below labelled 4.

4. (from Durrett)

(a) Let X and Y be independent random variables which take values in the integers. Prove that the distribution of $X+Y$ is given by

$$P(X+Y=n)=\sum_{m=-\infty}^{\infty} P(X=m)P(Y=n-m)$$

(b) X has a Poisson distribution with parameter $\lambda>0$ if it takes on the values $0,1,2,\cdots$ and

$$P(X=n)=\frac{e^{-\lambda}\lambda^n}{n!}$$

Show that if X and Y are independent random variables, X has a Poisson distribution with parameter λ and Y has a Poisson distribution with parameter μ, then $X+Y$ has a Poisson distribution. What is the parameter for $X+Y$?

4. (from Durrett) Let $\Omega=(0,1)$, $\mathcal{F}$ be the Borel sets in $(0,1)$, and let P be Lebesgue measure on $(0,1)$. For $n=1,2,\cdots$ define RV's by

$$X_n(\omega)=\begin{cases} 0 & \text{if } [2^n\omega] \text{ is even} \\ 1 & \text{if } [2^n\omega] \text{ is odd} \end{cases}$$

where $[x]$ is the largest integer less than or equal to x. Note that $0.X_1X_2X_3\cdots$ is the binary expansion of ω. We would like to prove that the $\{X_n\}_{n=1}^{\infty}$ are independent random variables. You are welcome to prove this, but writing it out can involve a lot of notation, so for this problem you need only prove that they are pairwise independent, i.e., for any $n\neq k$, the RV's X_n and X_k are independent. Note that this gives a rigorous construction of the probability space for flipping a fair coin infinitely many times.

The remaining problems are not to be turned in. They are included to make connections with the probability you see in a course like MATH/STAT 564 and cover some material from such a course that we do not cover.

The change of variables theorem says if X is a random variable and f a real-valued measurable function on the real line (with some condition on f), then

$$E[f(X)]=\int_{\mathbb{R}} f(x)\,d\mu_X$$

where μ_X is the distribution of X. The point of the next two problems is to get more explicit formulae that are amenable to computation in the cases that X is discrete or has a density.

5. Let X be a discrete real-valued random variable. Let $x_1, x_2, \cdots$ be its values and let $p_n = P(X = x_n)$. Let g be any real valued function on the real line. Suppose that

$$\sum_n |g(x_n)|\, p_n < \infty$$

Prove that $g(X)$ is a random variable and

$$E[g(X)] = \sum_n g(x_n) p_n$$

Note that I did not say that g was measurable.

6. Let X be a random variable which has a density $f(x)$. This means its distribution function $F(x)$ satisfies $F(x) = \int_{-\infty}^{x} f(u)\, du$, i.e., its distribution is $f(x)$ times Lebesgue measure. Let $g(x)$ be a real-valued measurable function on the real line such that

$$\int_{-\infty}^{\infty} |g(x)|\, f(x)\, dx < \infty$$

Prove that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

where the integrals with respect to dx are integration with respect to Lebesgue measure.

7. (564 discrete RV's) Let X be a discrete RV, $x_1, x_2, \cdots$ its values and $p_n = P(X = x_n)$. The function $f(x)$ which is p_n at x_n and is 0 at points not equal to one of the x_n is often called the "probability mass function" in 564 level probability courses.

Suppose we have a coin with probability p of heads. (p is not necessarily 1/2.) The following discrete RV's are of interest:

Binomial: Fix a positive integer n. Flip the coin n times and let X be the number of heads. So X can be $0, 1, 2, \cdots, n$.

Geometric: Flip the coin until you get heads for the first time. Let X be the number of tails. (Warning: depending on the author the definition of X is sometimes taken to be the total number of flips, including the final one that gave heads.) So the possible values of X are the nonnegative integers.

Negative binomial: Fix an integer r. Flip the coin until we get heads for the rth time. Let X be the total number of tails. So the possible values of X are the nonnegative integers.

For each of these find the probability mass function and the expected value of X.

8. (564 continuous RV's) In a 564 level course, "continuous RV" usually means that the distribution of the RV is absolutely continuous with respect to Lebesgue measure and so is given by $f(x)dx$. (Having a density implies the distribution function is continuous, but the converse is very false.) Here are three common "continuous" RV's.

Uniform: We say X is uniform on $[a,b]$ if $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and $f(x) = 0$ otherwise.

Exponential: We say X has an exponential distribution if $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Here λ is a positive parameter.

Normal: We say X has a normal distribution if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Here μ is a real parameter and σ^2 is a positive parameter.

Find the expected value of each of these random variables.

9. Let X be a random variable with a normal distribution. Find the density function of X^2. Hint: first find an expression for the distribution function of X^2.