Math 563 - Fall 21 - Homework 3

1. (Monte Carlo integration) Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$. Let $U_1, U_2, \cdots$ be independent and uniformly distributed on $[0,1]$, and let

$$I_n = \frac{1}{n}(f(U_1) + \cdots + f(U_n))$$

and let $I = \int_0^1 f(x)\,dx$. Prove that $I_n \to I_n$ a.s.

This is an easy problem. It is included since it is a very simple example of a numerical method called Monte Carlo integration. We can estimate the integral I by generating a large sample of uniformly distributed U_i and computing I_n. There are much better ways to compute 1-d integrals, but for very high dimensional integrals, Monte Carlo integration can sometimes be the best way to numerically compute the integral.

2. Suppose X_n are identically distributed, uncorrelated, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

(b) Now suppose that you also know that $EX_n^4 < \infty$ and the X_n are independent. Prove that $P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p.

3. (from Durrett) *metric for convergence in probability:* We look at the set of all random variables and consider two random variables to be the same if there are equal a.s. Define

$$d(X,Y) = E\left[\frac{|X-Y|}{|X-Y|+1}\right]$$

(a) Prove that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$.

(b) **Optional, will not be graded.** It is obvious that d is reflexive and $d(X,Y)=0$ if and only if $X=Y$ a.s. Prove that d satisfies the triangle inequality and so defines a metric.
(c) **Optional, will not be graded.** Prove that this metric space is complete.

4. Suppose we flip a fair coin infinitely many times. We say there is a run (of heads) of length n if somewhere in the sequence there are n heads in a row with a tails before them and a tails right after them. Let l be the length of the longest run in the sequence with $l=\infty$ if there are runs of arbitrarily long length. Prove that $P(l=\infty)=1$.

You can say a lot more about lengths of runs. Let l_n be the length of the longest run after n flips. Then you can get results about how l_n grows with n. This can be found in Durrett. This problem is asking you to do something simpler, so your proof should be simpler that what you can find in Durrett.

5. (from Durrett) Let X_n be independent with $P(X_n=1)=p_n$ and $P(X_n=0)=1-p_n$.
(a) Prove that $X_n \to 0$ in probability if and only if $p_n \to 0$.
(b) Prove that $X_n \to 0$ a.s. if and only if $\sum_n p_n < \infty$.