Math 563 - Fall 21 - Homework 4

1. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. (We also assume $\{X_n, Y_n : n = 1, 2, 3, \cdots\}$ is independent.) Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0, t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1] + E[Y_1]} \quad a.s.$$

2. Let X_n be an independent sequence of RV's.

(a) Prove that the RV's $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ are constant with probability one.

(b) Let $S_n = \sum_{k=1}^n X_k$. Prove that the following event has probability zero or one.

$$\{\omega : \frac{S_n}{n} \to 0\}$$

3. Convergence in distribution does not imply convergence in probability. But we do have the following result. Prove that if $X_n \Rightarrow c$ where c is a constant, then $X_n \to c$ in probability.

4. (from Durrett) Let X_n, X be integer valued RV's. Prove that $X_n \Rightarrow X$ if and only if for all integers m,

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

5. (from Durrett, converging together lemma) It is not true in general that if $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ then $X_n + Y_n \Rightarrow X + Y$. But we do have the following. If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant then $X_n + Y_n \Rightarrow X + c$. A useful consequence of this result is that if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X$.