Math 563 - Fall '21 - Homework 5

1. The gamma distribution is a two parameter family of densities. (There are different ways to parameterize it.) We take the density to be

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$$

for $x \geq 0$. The density is 0 for $x < 0$. The parameters β and α are both positive. Note that if $\alpha = 1$ we have the exponential distribution. The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

Integration by parts shows $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. If α is an integer then $\Gamma(\alpha) = (\alpha-1)!$. A little calculus shows the mean of the gamma distribution is α/β and the variance is α/β^2. You can also compute the characteristic function:

$$\phi(t) = \left(1 - \frac{it}{\beta}\right)^{-\alpha}$$

You do not have to include any of these calculations in your solution.

For positive integers n let X_n be a random variable with a gamma distribution with $\alpha = n$ and $\beta = 1$. Prove that $(X_n - n)/\sqrt{n}$ converges in distribution to a standard normal. There is more than one way to do this. One approach is to note that the sum of independent exponential distributions gives you a gamma distribution.

2. (Thinning a Poisson process) Let N_t be a Poisson process. Let X_n be an i.i.d. sequence which only takes on the values 0 and 1. Assume that the two sets of RV's $\{X_n\}_{n=1}^\infty$ and $\{N_t\}_{t\geq 0}$ are independent. Define a new process M_t by

$$M_t = |\{n \leq N_t : X_n = 0\}|$$

Here the absolute values just mean the cardinality of the set. Prove that M_t is a Poisson process.

3. (from Durrett) Let X_n be independent but not necessarily identically distributed. Let $S_n = \sum_{k=1}^n X_k$. Suppose $EX_n = 0$, $EX_n^2 = 1$ and there

are constants $0 < \delta, C < \infty$ such that $E|X_n|^{2+\delta} \leq C$ for all n. Prove that $S_n/\sqrt{n} \Rightarrow Z$ where Z has a standard normal distribution.

4. Let X_n be i.i.d. with $EX_n = 0$ and $EX_n^2 = 1$. Let $S_n = \sum_{k=1}^n X_k$. Use the central limit theorem and the Kolmogorov 0-1 law to prove that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty \quad a.s.$$

5. (based on Durrent 3.10.4) Let X_n and X be random vectors (values in $\mathbb{R}^d$). Suppose that X_n converges in distribution to X. Let $X_n = (X_n^1, \cdots, X_n^d)$ and $X = (X^1, \cdots, X^d)$.
(a) Prove that for each i the components X_n^i converge in distribution to X^i.
(b) Prove that the convergence of X_n^i in distribution to X^i for all the components i does not imply that X_n converges in distribution to X by giving a counterexample.