Math 563 - Fall '21 - Homework 6

1. (from Durrent)

(a) Let $X = (X_1, \cdots, X_d)$ be be a random vector. Let $\phi_X(t)$ be the characteristic function of X. (So it is a function on $\mathbb{R}^d$.) Let $\phi_{X_i}(t)$ be the characteristic function of X_i. (So it is a function on $\mathbb{R}$.) Prove that $X_1, X_2, \cdots, X_d$ are independent if and only if

$$\phi_X(t_1, t_2, \cdots, t_d) = \prod_{i=1}^{d} \phi_{X_i}(t_i)$$

(b) Let $X = (X_1, \cdots, X_d)$ have a multivariate normal distribution with covariance matrix C_{ij}. Prove that $X_1, X_2, \cdots, X_d$ are independent if and only the matrix C is diagonal.

2. (from Durrett) Let X_n be i.i.d. with a Poisson distribution with mean 1. Large deviation theory says

$$\lim_n \frac{1}{n} ln(P(S_n \geq an)) = \gamma(a)$$

for $a > 1$. Find the function $\gamma(a)$. It is also possible to do this without large deviation theory.

3. Let X and Y be independent random variables. X has a Poisson distribution with mean $E[X] = \lambda$ and Y has a Poisson distribution with mean $E[Y] = \mu$. Let $Z = X + Y$. Find $E[X|Z]$ and $E[Z|X]$. (One of these is easy.)

4. Let X be a random variable with $E|X| < \infty$. In general the σ-field generated by $|X|$ will be smaller than the σ-field generated by X. The goal of this problem is to find an explicit formula for $E[X|\sigma(|X|)]$ in a couple of special cases.

(a) Suppose X is a discrete RV (countable range). Find an explicit formula for $E[X|\sigma(|X|)]$.

(b) Now suppose the distribution of X is absolutely continuous with respect to Lebesgue measure and the density function is $f(x)$. Use your answer to part (a) to guess a formula for $E[X|\sigma(|X|)]$, and then show that your guess is correct. Your formula should involve f.