Math 563 - Fall '21 - Homework 7

1. (from Durrett) Define $var[X|\mathcal{F}]$ to be $E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$. Prove that

$$var(X) = E[var[X|\mathcal{F}]] + var(E[X|\mathcal{F}])$$

2. Let ξ_n be an i.i.d. sequence and suppose that $E|\xi_n|^2 < \infty$. Let $X_n = \sum_{i=1}^n \xi_i$. Show that there are constants a, b, c, d such that
(a) $X_n + na$ is a martingale
(b) $X_n^2 + bnX_n + cn^2 + dn$ is a martingale.

3. Let X_n be a discrete time Markov chain with transition matrix $p(x, y)$. So it takes values in a countable or finite set S. Let $f : S \to \mathbb{R}$ and define a function Pf on S on

$$Pf(x) = \sum_y p(x, y) f(y)$$

So P is a linear operator on the space of functions on S. We say that f is P-harmonic if
(i) $\sum_y p(x, y)|f(y)| < \infty, \quad \forall x \in S$
(ii) For all $x \in S$, $f(x) = Pf(x)$.

Prove that if f is P-harmonic, then $f(X_n)$ is a martingale.