Math 563 - Fall '21 - Midterm (take home)

Do 4 of the 5 problems. Do not turn in more than 4.

Due Wednesday, Oct 27, 11:59 pm - no late papers accepted

The fine print: You are supposed to do this exam on your own. This means you should not talk to anyone about the exam. You can ask me questions, but I will only answer questions about what the problem is asking, not about how to do it. You can consult your class notes, past homeworks and Durrett. It is probably possible to find the solution to every one of these problems somewhere on the web. Don't do this!

1. Let X_n be an independent, identically distributed sequence of non-negative random variables. Prove that $EX_1 < \infty$ if and only if $P(X_n \geq n \ i.o.) = 0$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let A be an event with $P(A) > 0$. Define a new probability measure on $(\Omega, \mathcal{F})$ by $Q(B) = P(A \cap B)/P(A)$. (So $Q(B) = P(B|A)$.) Let X be a nonnegative random variable which is independent of A, meaning that $\sigma(X)$ and $\sigma(A)$ are independent. ($\sigma(A)$ is just $\{A, A^c, \Omega, \emptyset\}$.) Prove that

$$\int X dP = \int X dQ \tag{1}$$

Hint: start with a simple random variable X.

3. Let X_n be an i.i.d. sequence with zero mean and finite variance. Prove that

$$\frac{\sum_{k=1}^n X_k}{(\sum_{k=1}^n X_k^2)^{1/2}}$$

converges in distribution to a standard normal. Hint: what does the strong law of large numbers tell you about the denominator?

4. The random variables X_n, Y_n, X, Y are all defined on the same probability space. Suppose that X_n converges to X in distribution and that Y_n converges to Y in distribution. Suppose also that X and Y are independent and that for each n the RV's X_n and Y_n are independent. Prove that $X_n + Y_n$ converges in distribution to $X + Y$. Hint: characteristic functions.

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose that Ω is countable. Let X_n be a sequence of random variables. Prove that $X_n \to 0$ a.s. if and only if $X_n \to 0$ in probability.