Math 565C - Spring 24 - Homework 1
Due Mon, Jan 29

1. The point of this problem is to show that there are at least three different ways to define Brownian motion. We only consider one-dimensional BM started at 0. Let B_t be a stochastic process on $[0,\infty)$. Prove that the following three statements are equivalent. Note that several parts of this equivalence were proved in class. For those parts you can just say that it was proved in class. You do not need to repeat what was done in class.
(A) The finite dimensional distributions of B_t are given by

$$P(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k) =$$
$$\int_{F_1} \int_{F_2} \cdots \int_{F_k} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_k \cdots dx_2 dx_1$$

where $t_1 < t_2 < \cdots t_k$ and

$$p(t,x,y) = (2\pi)^{-1/2} \exp(-\frac{|x-y|^2}{2t})$$

(This was the definition of Brownian motion from class.)
(B) B_t is a Gaussian process with mean $m(t) = 0$ and covariance $C(s,t) = \min\{s,t\}$.
(C) B_t has independent increments and for $0 \leq s < t$, $B_t - B_s$ has a normal distribution with mean zero and variance $t - s$.

2. Do one of the following two problems
(a) Let B_t be a one-dimensional Brownian motion started at 0. Let $c > 0$ and define

$$\hat{B}_t = c B_{c^{-2} t}$$

Prove that B_t is a Brownian motion
(b) Define

$$\hat{B}_t = t B_{1/t}$$

for $t > 0$ and define $\hat{B}_0 = 0$. Prove that $\hat{B}_t$ is a Browian motion

3. In class I sketched a construction of Brownian motion using the Haar basis. The point of this problem is to fill in some of the proof I did not do. Recall the functions $S_{n,k}(t)$ on $[0,1]$ were continuous with support in

$[k2^{-n}, (k+1)2^{-n}]$ where $n = 0, 1, 2, \cdots$ and $k = 0, 1, ...2^n - 1$. The max of $S_{n,k}$ is $2^{-n/2-1}$. Brownian motion on $[0,1]$ was defined to be

$$B_t = \sum_{n,k} Z_{n,k}\, S_{n,k}(t)$$

where the $Z_{n,k}$ are independent random variables each of which has a standard normal distribution. This problem will show that with probability one this series converges uniformly absolutely. (This implies B_t is continuous on $[0,1]$ with probability one.)
(a) Suppose $a_{n,k}$ are constants and there exists ϵ with $0 < \epsilon < 1/2$ and $C < \infty$ such that

$$|a_{n,k}| \le C 2^{n\epsilon}$$

for all n, k. Prove that

$$\sum_{n,k} a_{n,k} S_{n,k}$$

converges uniformly absolutely.
(b) Fix an ϵ with $0 < \epsilon < 1/2$. Let E be the event that there exists a random constant M such that for all n

$$\max_{2^n \le k < 2^{n+1}} |Z_{n,k}| \le M 2^{n\epsilon}$$

Prove that $P(E) = 1$. Hint : use the "easy" half of the Borel-Cantelli lemma.

4. Let B_t be a Brownian motion. **Using just the definition of the Ito integral,** show that

$$\int_0^t B_s^2\, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s\, ds$$

You cannot use the Ito formula in your solution. (The problem is trivial if you use the Ito formula.) This is problem 3.2 in Oksendal. For a similiar but easier problem with a solution see his problem 3.1.

5. (problem 3.7 in Oksendal) A result of Ito says

$$n! \int_0^t \left(\int_0^{s_n} \left(\int_0^{s_{n-1}} \cdots \left(\int_0^{s_3} \left(\int_0^{s_2} dB_{s_1} \right) dB_{s_2} \right) \cdots dB_{s_{n-2}} \right) dB_{s_{n-1}} \right) dB_{s_n}$$
$$= t^{n/2} h_n\left(\frac{B_t}{\sqrt{t}}\right)$$

where $h_n(x)$ is the nth Hermite polynomial. In particular $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$.

(a) Verify that in each of the n Ito integrals the integrand satisfies the requirements in the definiton of the Ito integral.

(b) Verify the above formula for $n = 1, 2, 3$. Hint: use problem 4 above and example 3.1.9 in Oksendal.

For a challenge, prove the result for all n. This is optional.