Math 565C - Spring 24 - Homework 2
Due ???

1. Let B_t be Brownian motion. Let σ be a constant. For $t \geq 0$ let

$$M_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$$

In class I showed (or will show) that M_t is a martingale using the Ito formula. Prove that M_t is a martingale just using the definition of a martingale and properties of Brownian motion.

2. (Exercise 3.10 in Oksendal) Let $f(t,\omega) \in \mathcal{V}(0,T)$. Suppose that there exist constants $K < \infty$ and $\epsilon > 0$ such that

$$E[|f(s,\omega) - f(t,\omega)|^2] \leq K|t-s|^{1+\epsilon}, \quad s,t \in [0,T]$$

This is a form of "smoothness" of f as a function of t.

Let $\{t_j^{(n)}\}$ be a sequence of partitions such that the largest gap in the nth partition goes to zero as $n \to \infty$. For each partition let $\bar{t}_j^{(n)} \in [t_j^{(n)}, t_{j+1}^{(n)})$ Prove that

$$\sum_j f(\bar{t}_j^{(n)}, \omega)(B_{t_{j+1}} - B_{t_j})$$

converges to the Ito integral $\int_0^T f(t,\omega)\,dB_t$ as $n \to \infty$. The Stratonovich integral is defined by taking $\bar{t}_j^{(n)}$ to be the midpoint $(t_j^{(n)} + t_{j+1}^{(n)})/2$. So this result shows that for such an f the Ito integral is equal to the Stratonovich integral.

3. (Exercise 3.13 in Oksendal)

(a) A stochastic process X_t is continuous in mean square if for all t we have $E[X_t^2] < \infty$ and

$$\lim_{s \to t} E[(X_s - X_t)^2] = 0$$

For example, Brownian motion is continuous in mean square.

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. This means that there is a constant C such that

$$|f(x) - f(y)| \leq C|x-y|, \quad \forall x, y$$

Prove that if X_t is continuous in mean square then $f(X_t)$ is continuous in mean square. (This is easy.) In particular this shows that any Lipschitz function of Brownian motion is continuous in mean square.

(b) Let X_t be a stochastic process which is continuous in mean square. Fix $T < \infty$. Suppose that $X_t(\omega) \in \mathcal{V}(0,T)$ so that the Ito integral of $X_t(\omega)$ is defined. Prove that

$$\int_0^T X_t \, dB_t = \lim_{n \to \infty} \int_0^T \phi_n(t,\omega) \, dB_t$$

where

$$\phi_n(t,\omega) = \sum_j X_{t_j}(\omega) 1_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)$$

and $\{t_j^{(n)}\}$ is a sequence of partitions such that the largest gap in the nth partition goes to zero as $n \to \infty$.

4. Define $h_n(t,x)$ by

$$h_n(x,t) = \frac{(-t)^n}{n!} \exp(\frac{x^2}{2t}) \frac{d^n}{dx^n} \exp(-\frac{x^2}{2t})$$

(These functions are related to the Hermite polynomials.) Prove that

$$dh_{n+1}(B_t,t) = h_n(B_t,t) dB_t$$

In particular this shows that $h_{n+1}(B_t,t)$ is a martingale.

More examples of the use of the Ito lemma can be found in exercises 4.1,4.2,4.3 (product rule), 4.11 in Oksendal, all of which have solutions in the book.

5. (Oksendal 4.13) Let X_t be the Ito process given by $dX_t = u(t,\omega)dt + dB_t$. Assume that u is bounded. X_t is not a martingale unless $u = 0$. We can modify X_t to create a martingale as follows. Define

$$Y_t = X_t \exp\left(- \int_0^t u(s,\omega) \, dB_s - \frac{1}{2} \int_0^t u^2(s,\omega) \, ds \right)$$

Use Ito's formula to prove that Y_t is a martingale with respect to the σ-field generated by B_t.