Math 565C - Spring 24 - Homework 3
Due Fri, March 1

1. (Evans problem 41) Let X_t solve the Langevin equation with the initial condition that X_0 is normal with mean 0 and variance $\frac{\sigma^2}{2b}$ and independent of the Brownian motion. We saw in class that X_t is a Gaussian process with mean zero. Find the covariance of this process. The Langevin equation is discussed in Evans pp. 86-88.

2. (Oksendal 5.7) The mean-reverting Orstein Uhlenbeck process is given by

$$dX_t = (m - X_t)dt + \sigma dB_t$$

where m, σ are constants.
(a) Solve this equation
(b) Find $E[X_t]$ and $var(X_t)$. For more fun you can find the covariance of the process, but this is optional.

3. (Evans problem 43) One approach to trying to solve an SDE is to assume the solution has the form $X_t = g(t, B_t)$. (This assumption is not valid in general, but for some SDE's it works.)
(a) Use this method to solve

$$\begin{aligned} dX_t &= -\frac{1}{2} e^{-2X_t}\, dt + e^{-X_t}\, dB_t, \\ X(0) &= x_0 \end{aligned}$$

where x_0 is a constant.
(b) Show that the solution blows up at a finite random time. Explain (very briefly) why this does not contradict our existence theorem which give existence of a solution for all time.

4. (Oksendal 4.13) Let X_t be an Ito process of the form

$$dX_t = u(t,\omega)dt + dB_t$$

This is a martingale only if $u = 0$. When u is nonzero we can get a martingale by multiplying X_t by a suitable exponential. Assume that u is bounded and define $Y_t = X_t M_t$ where

$$M_t = \exp\left(-\int_0^t u(s,\omega)dB_s - \frac{1}{2}\int_0^t u^2(s,\omega)\,ds\right)$$

Use Ito's formula to prove that Y_t is a martingale with respect to the Brownian filtration. See Oksendal for a comment on how this is related to the Girsanov theorem.

5. (Oksendal 5.14) Complex Brownian motion $B(t)$ is defined as

$$B(t) = B_1(t) + iB_2(t)$$

where B_1, B_2 are independent (real-valued) Brownian motions.

(a) Let $F(z)$ be an analytic function. Define $Z_t = F(B(t))$. Show that $dZ_t = F'(B(t))\,dB(t)$. (Here $F'(z)$ is the usual complex derivative of $F(z)$.) Z_t is a complex valued process. You can compute dZ_t by applying Ito's lemma to its real and imaginary parts. Hint: Cauchy-Riemann equations.

(b) Solve the SDE

$$dZ_t = \alpha Z_t\,dB(t)$$

where α is a constant.