Math 565C - Spring 24 - Homework 4
Due Wed, March 27

1. (Oksendal problem 7.3) Let B_t be 1d Brownian motion with $B_0 = 0$. Define

$$X_t = exp(ct + \alpha B_t)$$

where c, α are constants. Prove directly from the definition that X_t is a Markov process, i.e., that

$$E[f(X_{t+h})|\mathcal{F}_t] = E^{X_t}[f(X_h)]$$

for bounded Borel-measurable f.

2. (based on Oksendal problem 7.8)

(a) Prove that if τ_1 and τ_2 are stopping times, then $\min\{\tau_1, \tau_2\}$ and $\max\{\tau_1, \tau_2\}$ are stopping times.

(b) Let τ_n be a decreasing sequence of stopping times. Prove that $\tau = \lim_n \tau_n$ is a stopping time.

(c) Let F be closed in $\mathbb{R}^n$. Let X_t be an Ito diffusion in $\mathbb{R}^n$ with continuous paths a.s. Let τ_F be the first exit time for F, i.e.,

$$\tau = \inf\{t : X_t \notin F\}$$

Prove that τ is a stopping time. Hint: because the paths are continuous and F is closed, you can express τ in terms of X_t at just the rational t's.
Optional: Prove the same result with F replaced by an open set. Hint: An open set can be written as the union of an increasing sequence of closed sets.

3. (Oksendal problem 7.10) Let X_t be the geometric Brownian motion

$$dX_t = rX_t dt + \alpha X_t dB_t$$

Let $T > 0$ be a constant time and $t < T$. The goal in this problem is to compute

$$E^x[X_T|\mathcal{F}_t] \tag{1}$$

in two different ways.

(a) Compute it using the Markov property.

(b) You can solve this SDE explicitly and then write the solution as

$$X_t = xe^{rt}M_t$$

where

$$M_t = \exp(\alpha B_t - \frac{1}{2}\alpha^2 t)$$

We know that M_t is a martingale. (You do not need to show this.) Use this to compute (1) in a second way.

4. Consider the SDE

$$dX_t = a(t)X_t dt + b(t)X_t \circ dB_t$$

where the $\circ$ means that we interpret this SDE in the Stratonovich sense. Solve this SDE with initial condition X_0 where X_0 can be random but is independent of the Brownian motion. Recall that we solved this SDE when it is interpreted in the Ito sense. (This problem should not be hard.)

5. Suppose that $b(t,x)$ and $\sigma(t,x)$ satisfy the hypotheses of the existence/(strong) uniqueness theorem. Suppose that $X_t, \bar{X}_t$ are adapted processes, both of which are continuous in mean square (recall exercise 3 in homework 2). Define

$$\begin{aligned} Y_t &= \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s \\ \bar{Y}_t &= \int_0^t b(s, \bar{X}_s) ds + \int_0^t \sigma(s, \bar{X}_s) d\bar{B}_s \end{aligned}$$

where B_t and $\bar{B}_t$ are each Brownian motions.

(a) Prove that the processes Y_t and $\bar{Y}_t$ have the same distribution.

(b) Prove that Y_t is continuous in mean square.

(c) Now consider the Ito diffusions

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ d\bar{X}_t &= b(t, \bar{X}_t)dt + \sigma(t, \bar{X}_t)d\bar{B}_t \end{aligned}$$

with initial conditions X_0 and $\bar{X}_0$ where X_0 and $\bar{X}_0$ have the same distribution. Let X_t^k and $\bar{X}_t^k$ be the sequence of approximations to X_t and $\bar{X}_t$ that come from Picard iteration. Use (a) and (b) to prove that the processes X_t^k and $\bar{X}_t^k$ have the same distribution for all k and that they are continuous in mean square for all k. This is easy given (a) and (b).

(d) Conclude that the proceeses X_t and $\bar{X}_t$ have the same distribution, i.e., we have weak uniqueness. Can we conclude that the solutions X_t and $\bar{X}_t$ are continuous in mean square?