Math 565C - Spring 24 - Homework 5
Due Wed, April 10

1. (Oksendal problem 7.3) Let B_t be 1d Brownian motion with $B_0 = 0$. Define

$$X_t = x\, exp(ct + \alpha B_t)$$

where c, α are constants and x is non-random. Prove directly from the definition that X_t is a Markov process, i.e., that

$$E[f(X_{t+h})|\mathcal{F}_t] = E^{X_t}[f(X_h)]$$

for bounded Borel-measurable f.

2. Let X_t be the one-dimensional diffusion

$$dX_t = b\, dt + dB_t$$

with $b \neq 0$. This is just Brownian motion with a drift: $X_t = bt + B_t$. Fix $a > 0$. Start the process at $X_0 = x_0$ with $0 < x_0 < a$. Let τ be the time when the process exits $(0, a)$, i.e.,

$$\tau = \inf\{t > 0 : X_t \notin (0, a)\}$$

(a) Prove that $E[\tau] < \infty$
(b) Compute $P(X_\tau = a)$ and $E[\tau]$.

Hint: In class we did this for $b = 0$ by using the Dynkin formula.

3. Oksendal 7.9

4. Oksendal 8.2

5. Oksendal 8.3