Math 565a - Homework 1

1. Let $\{B_k\}_{k=1}^\infty$ be events which form a countable partition of Ω. We assume that $P(B_k) > 0$ for all k. This means that the events are disjoint and their union is all of Ω. Let $\mathcal{G}$ be the σ algebra generated by $\{B_k\}_{k=1}^\infty$. Let X be an integrable random variable and define

$$E[X|B_k] = \frac{E[X; B_k]}{P(B_k)}$$

Prove that

$$E[X|\mathcal{G}] = \sum_{k=1}^\infty E[X|B_k] \, 1_{B_k} \quad a.s.$$

2. The goal of this problem is to connect our undergraduate notion of conditional expectation with the one we have studied in class. Let X, Y be random variables that have a joint probability density function $f_{X,Y}(x, y)$. So $f_{X,Y}(x, y)$ is an integrable function on $\mathbb{R}^2$ with respect to Lebesgue measure and

$$P((X, Y) \in B) = \int \int 1((x, y) \in B) \, f_{X,Y}(x, y) \, dxdy$$

for Borel subsets B of $\mathbb{R}^2$. One can show that for a function $g(x, y)$ such that $g(X, Y)$ is an integrable random variable, we have

$$E[g(X, Y)] = \int \int g(x, y) \, f_{X,Y}(x, y) \, dxdy$$

(You may use this fact.) The conditional pdf of X given Y is defined as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

when $f_Y(y) > 0$. When $f_Y(y) = 0$ we make the convention that $f_{X|Y}(x|y) = 0$. Here $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx$ is the marginal pdf of Y. Let h be a Borel measurable function on $\mathbb{R}$ such that $h(X)$ is integrable. Define

$$g(y) = \int h(x) \, f_{X|Y}(x|y) \, dx$$

Prove that $E[h(X)|Y] = g(Y) \quad a.s.$

3. If X is an integrable random variable, then given $\epsilon > 0$ we can find M so that $E[|X|; |X| > M] < \epsilon$. Now let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of random variables. The index set $\mathcal{A}$ can be uncountable. We say the collection is *uniformly integrable* if $\forall \epsilon > 0$ there exists an M such that
$E[|X_\alpha|; |X_\alpha| > M] < \epsilon$ for all α. Now fix an integrable random variable X and consider the collection of random variables $E[X|\mathcal{G}]$ where $\mathcal{G}$ ranges over all sub σ algebras of $\mathcal{F}$. Show this collection is uniformly integrable.

4. (Durrett) Let $\mathcal{G}$ be a sub σ algebra. Let $B \in \mathcal{G}$. Show that if X and Y are integrable random variables which agree on B a.s., then $E[X|\mathcal{G}] = E[Y|\mathcal{G}]$ a.s. on B. Hint: mimic the proof that if $X \geq 0$, then $E[X|\mathcal{G}] \geq 0$.

5. (Durrett) Let X, Y be random variables with $E|X| < \infty$,$E|Y| < \infty$, and $E|XY| < \infty$. Consider the following three statements.
(1) X and Y are independent
(2) $E[Y|X] = E[Y] \quad a.s.$
(3) $E[XY] = E[X]E[Y]$
We know that (1) implies (3) and that (1) implies (2). Show that (2) implies (3). Give counterexamples to show that (2) does not imply (1) and (3) does not imply (2). You can find counterexamples in which X and Y only take on the values $-1, 0, 1$.

6. Fix a filtration $\mathcal{F}_n$. Let τ_k be a sequence of stopping times.
(a) Prove that $\sup_k \tau_k$, $\inf_k \tau_k$, $\liminf_{k\to\infty} \tau_k$, and $\limsup_{k\to\infty} \tau_k$ are all stopping times.
(b) Is the sum of two stopping times a stopping time? You should either prove that it is or give a counterexample.

7. Let τ be a stopping time for the filtration $\mathcal{F}_n$. Recall that $\mathcal{F}_\tau$ is defined to be the collection of events A such that $A \cap \{\tau \leq n\} \in \mathcal{F}_n$ for all n.
(a) Show that $\mathcal{F}_\tau$ is a σ algebra.
(b) If X_n is a stochastic process adapted to $\mathcal{F}_n$, then we can use the stopping time τ to define a new process which is the process stopped at the stopping time: $X_n^\tau = X_{\min\{n,\tau\}}$. Show that X_n^τ is adapted with respect to $\mathcal{F}_n$.
(c) Define $\mathcal{F}_n^\tau = \mathcal{F}_{\min\{\tau,n\}}$. (Note that $\min\{\tau, n\}$ is a stopping time, so $\mathcal{F}_{\min\{\tau,n\}}$ is defined by the definition at the start of this problem.) Show that $\mathcal{F}_n^\tau$ is a filtration and X_n^τ is adapted with respect to this filtration.