Math 565a - Homework 2

1. Consider a one-dimensional random walk whose steps have mean zero. (We do not assume it is simple.) Let $c > 0$ and define a stopping time by $\tau = \inf\{n : S_n \geq c\}$. Prove that $E\tau = \infty$.

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3. Suppose that the random walk $\{S_n\}_{n=1}^{\infty}$ is recurrent. Fix a positive integer k. By $\{S_{nk}\}_{n=1}^{\infty}$ we mean the random walk we get by only looking at where S_n is every k time steps. So the steps of the random walk $\{S_{nk}\}_{n=1}^{\infty}$ have the same distribution as $X_1 + \cdots + X_k$. Prove that $\{S_{nk}\}_{n=1}^{\infty}$ is recurrent. Hint: Prove that if $\{S_{nk}\}_{n=1}^{\infty}$ is transient, then $\{S_{nk+j}\}_{n=1}^{\infty}$ is transient for any positive integer j. You may not use the theorem stated (but not proved) in class that gives a necessary and sufficient condition for recurrence that involved the charateristic function.

4. Consider a simple random walk in one-dimension. So $X_k = \pm 1$. Let a, b be integers with $a < 0 < b$. Let $\tau = \inf\{n : S_n = a \quad or \quad b\}$. Prove that $P(\tau = n)$ decays exponentially with n and so $E\tau < \infty$. Hint: If $X_k = +1$ for $k = l, l+1, \cdots, l+b-a$, then $\tau \leq l + b - a$.

5. In this problem we consider the recurrence/transience of the simple symmetric random walk in one and two dimensions. It follows immediately from theorems we proved in class that it is recurrent for $d = 1, 2$. The point of this problem is to show it is recurrent without using these theorems by showing $\sum_{n=1}^{\infty} P(||S_n|| < \epsilon) = \infty$ for some $\epsilon > 0$. If $\epsilon < 1$, then $||S_n||_\infty < \epsilon$ is equivalent to $S_n = 0$. So we consider the sum $\sum_{n=1}^{\infty} P(S_n = 0)$.
(a) In $d = 1$ there is a simple expression for $P(S_n = 0)$. Use Stirlings formula to determine how it behaves for large n and show $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$.
(b) Let $\rho_d(n)$ denote $P(S_n = 0)$ for the d dimensional simple symmetric random walk. Show that $\rho_2(n) = \rho_1(n)^2$, and use this to show the two-dimensional walk is recurrent. Hint: One approach is to find an explicit formula for $\rho_2(n)$ and manipulate it to show it equals $\rho_1(n)^2$. Another approach is to rotate the lattice $\mathbb{Z}^2$ and think about the distribution of the steps of the simple random walk in this rotated coordinate system.

6. Suppose that the set of possible values is discrete. Let τ_n be the time of the nth return to the origin. When the walk does not return to the origin at least n times, τ_n is defined to be ∞. Show

$$P(\tau_n < \infty) = P(\tau_1 < \infty)^n$$

7. Let τ_n be stopping times that are finite a.s. with $n = 1, 2, 3, \cdots$. Assume that $\tau_n \leq \tau_{n+1}$ a.s. for $n = 1, 2, 3, \cdots$. Let S_n be a random walk with values in $\mathbb{R}^d$. Prove that $S_{\tau_1}, S_{\tau_2} - S_{\tau_1}, S_{\tau_3} - S_{\tau_2}, ...$ are independent.